COMP2610/6261 - Information Theory

Lecture 17: Lempel-Ziv Coding and Summary

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Eliminating Repetition

What is a simple, short binary description of the following string?

$$\underbrace{\texttt{aaaaaaa}}_{7\texttt{a}s}\ \underbrace{\texttt{bbbbbbb}}_{7\texttt{b}s}\ \underbrace{\texttt{aaaaaaa}}_{7\texttt{a}s}\ \underbrace{\texttt{bbbbbbb}}_{7\texttt{b}s}$$

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Run-length coding using (count, symbol) saves 12 bits:

$$\underbrace{111}_{7}\underbrace{0}_{\texttt{a}}\ \underbrace{111}_{7}\underbrace{1}_{\texttt{b}}\ \underbrace{111}_{7}\underbrace{0}_{\texttt{a}}\ \underbrace{111}_{7}\underbrace{1}_{\texttt{b}}$$

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- Makes no probabilistic assumptions about source.
- Doesn't always yield shorter strings:
  aa bb a b a $\rightarrow$ 10 0 10 1 01 0 01 1 01 0 (7 to 15 bits)
- Misses other structure: "2 repetitions of (7 as and 7 bs)"

Looking for Repetition

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$$\texttt{abbababbababbab}\ldots$$

We can describe each new part in terms of what we have seen so far:

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`abbababbababbab...`

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2. A new `b`
3. The same 1 symbol as 1 symbol ago

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Consider a sequence that starts

$$\underline{\mathtt{ab}}\mathtt{b}\color{red}{\mathtt{ab}}\color{black}{\mathtt{abbababbab}\ldots}$$

We can describe each new part in terms of what we have seen so far:

1. A new a
2. A new b
3. The same 1 symbol as 1 symbol ago
4. The same 2 symbols as 3 symbols ago

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5. The same 10 symbols as 5 symbols ago

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Consider a sequence that starts

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We can describe each new part in terms of what we have seen so far:

1. A new a $(0, \texttt{a})$
2. A new b $(0, \texttt{b})$
3. The same 1 symbol as 1 symbol ago $(1, 1, 1)$
4. The same 2 symbols as 3 symbols ago $(1, 3, 2)$
5. The same 10 symbols as 5 symbols ago $(1, 5, 10)$

00 01 10010001 10110010 11011001 ...

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LZ77(Sequence $x_1 x_2 \ldots$, Window size $W > 0$)

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      3. Output $(1, i, |s|)$

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Notes:

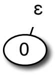
- The output is converted to binary. i is represented with $\lceil \log_2 W \rceil$ bits.
- The size output $|s|$ can be larger than W.
- Not very effective compression for short input sequences.
- Run-length encoding is essentially LZ77 with $W = 1$.

Extending Substrings

Consider the same sequence as before

abbababbababbab . . .

Scan sequence and record each previously unseen string:



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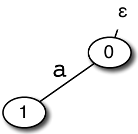

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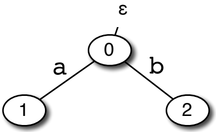

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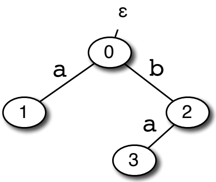
3. ba $(2, \texttt{a})$



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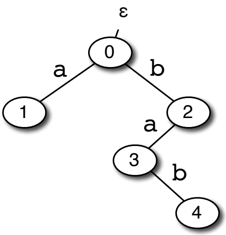

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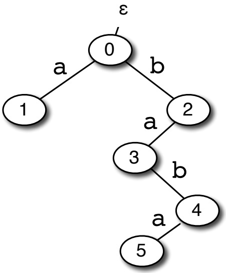

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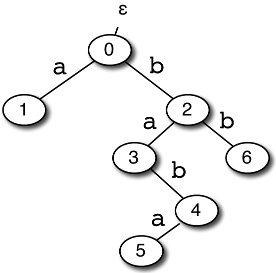

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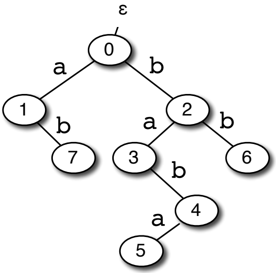
6. bb — $(2, \texttt{b})$
7. ab — $(1, \texttt{b})$



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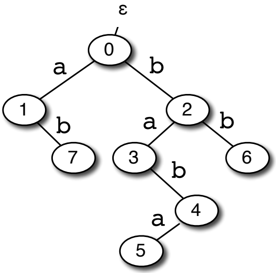

$$0\,0\;\;0\,1\;\;10\,0\;\;11\,1\;\;100\,0\;\;010\,1\;\;001\,1\ldots$$

Lempel-Ziv: Tree-Structured (LZ78)

See MacKay §6.4

LZ78(Sequence $x_1 x_2 \ldots$)

1. Initialise empty dictionary $D \leftarrow \{\}$ and $s \leftarrow \epsilon$

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Notes:

- Only $\lceil \log_2 n \rceil$ bits of i need to be output at step n
- $|D|$ is the number of entries in the dictionary
- Basic algorithm has several inefficiencies (e.g., if aa and ab in dictionary, then a is not needed)
- Decoding via "identical twin"

Theory

- Both LZ77 and LZ78 are *optimal* in the sense that the expected bits per symbol from some source X converges to $H(X)$ as $N \to \infty$
- Proofs are involved and not covered in this course (See Cover & Thomas §13.5)

Lempel-Ziv: Theory and Practice

Theory

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Practice

- LZ77 forms the basis of `gzip`, WinZip, the PNG image format, as well as PDF and HTTP compression.
- Variants of LZ78 (a.k.a. LZW) are used for the GIF image format, UNIX `compress`, and early modem protocols.
- Run-Length Encoding (RLE), along with the *Burrows-Wheeler Transform* (BWT) and Huffman coding are used in the `bzip2` compressor

Summary and Reading

Summary:

- Run-length Encoding
- Lempel-Ziv Coding
  - Sliding Window (LZ77)
  - Tree-Structured (LZ78)

Reading

- MacKay §6.4
- Cover & Thomas §13.4

Compression: Review

Block Coding (Uniform Length)

- Lossy compression
- Typical Sets
- Source Coding Theorem

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Source Coding Theorem for Block Codes (Lecture 11)

For all $\delta \in (0, 1)$ and $\epsilon > 0$ there is an N_0 such that for all $N > N_0$

$$\left| \frac{1}{N} H_\delta \left(X^N \right) - H(X) \right| < \epsilon$$

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Symbol Coding (Variable Length)

- Kraft inequalities: Unique decodability limits compression
- Source Coding Theorem: Witnessed by Shannon Codes
- Huffman Coding: Optimal Symbol Coding
- Shannon-Fano-Elias Coding: Codes from Intervals

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Kraft Inequality (Lecture 13)

The code lengths $\{\ell_1, \ldots, \ell_I\}$ for an alphabet of I symbols are for a uniquely decodable code if and only if $\sum_i 2^{-\ell_i} \leq 1$

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Source Coding Theorem for Symbol Codes (Lecture 14)

For any ensemble X there exists a code C (the *Shannon Code*) such that

$$H(X) \leq L(C, X) \leq H(X) + 1$$

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Stream Coding (Dynamic Codes)

- Arithmetic Coding: Probabilistic Models – e.g., Dirichlet
- Lempel-Ziv Coding: Model-free

The Limits of Compression

In each of the types of compression schemes the goal is to compress down to the entropy of the source.

Q: Why can't I repeatedly compress a message to make it smaller?

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Next Time:

Error Correction – Putting the redundancy back in!