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DEPARTMENT OF PURE MATHEMATICS

HARMONIC MEASURE ON RADIALLY SLIT DISKS

by

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A thesis submitted for consideration in the degree of Bachelor of Science with honours in pure mathematics at the University of New South Wales.
November 1996

Acknowledgements

First and foremost a great deal of thanks must go to my supervisor, Dr. Byron Walden. His focused guidance and constant support, especially around the completion of this thesis, is what has made this manuscript possible.

I would also like to express my gratitude to the rest of the academic and general staff in the School of Mathematics. Special thanks must go to Professor Colin Sutherland, Associate Professor Tony Dooley, Dr. Ian Doust, Dr. Norman Wildberger, and Dr. Hendrik Grundling for making my Honours year a fascinating and memorable one.

I am grateful to Dr. Randolf Bank, the author of `PLTMG`, who assisted me via e-mail as I was learning how to use his software. Thanks also go to Dr. Quine for his personal correspondence.

My flat-mates throughout the year, Jarrod, Julie, Trichelle, Shane and Elise deserve credit for putting up with me and room, which is generally in a state of disarray. Thanks go to my fellow students Bill, Stu, Lyria, Brendon, Andrew, Colin, Evan, Jason Phil and Eddie for their friendship and excellent lecture-note taking abilities. Simon and Dmitri are singled out and thanked separately for finishing their Honours year with me.

Throughout the course of this degree my family have provided me with an enormous amount of support both emotionally and financially. The dedication of this thesis to them is a token of my immeasurable thanks.

Contents

Chapter 1

Introduction

Geometric function theory is a branch of mathematics that uses analytic properties of functions to deduce geometric properties about domains, and vice-versa. This interplay of ideas can be seen in many principles used in classical analysis, especially of functions of a complex variable. Questions that arise in this field are often extremal in nature, involving finding functions or domains for which certain properties of either are minimised or maximised. A famous example of such a problem is that of Milloux's. Suppose we have a set in the unit disk that intersects each circle in the disk once. At each of these intersections it is known that a function takes its minimal value for that circle there. What then, is the largest values this function can take at a given point in the disk?

Instrumental in the solution to this and other such problems is the notion of harmonic measure. This conformal invariant can be thought of as a function on the domain that, at each point, assigns weights to subsets of the boundary. Intuitively, the weights reflect how much of these subsets can be "seen" from the given point. A theorem of Kakutani's [13] best describes this "seeing" by showing that harmonic measure and hitting probability of a Brownian motion are one and the same.

The Milloux problem can be solved by finding the arrangement of the minimal set that minimises its harmonic measure at the point in question. Beurling showed (see for example [1]) that this arrangement is the one formed by "hiding" all the points in the set behind one another in a radial slit. The question motivating the material presented in this thesis is a natural extension of this problem: Given a collection of radial slits in the disk, how can they be arranged so as to maximise the harmonic measure taken at the origin? It was conjectured by Gončar that this maximal arrangement would be when the slits are evenly spaced within the disk. Proving this statement has been met with some success. In 1984, Dubinin [8] demonstrated, using his notion of *desymmetrization*, that when the radially slit disk is simply connected Gončar's conjecture is true. Also used in his proof is the older, comparable theory of symmetrization due to Pólya and Szegö.

The case when the domain is multiply connected is still an open question. Baernstein [5] has, however, generalised Gončar's conjecture to a stronger statement involving integral means of harmonic measures and proven this extension for multiply connected domains with up to three slits.

This thesis presents the theorems of Dubinin and Baernstein along with the relevant background material on harmonic measure and other tools needed to supply the proofs. Chapter 1 introduces harmonic measure and some of its properties as well as some useful results from the theory of conformal maps and the Dirichlet problem. In Chapter 2, the parallel theories of symmetrization and desymmetrization are presented along with a powerful way of reinterpreting integral means due to Baernstein. The proofs of the theorems themselves follow readily in Chapter 3, and in the final chapter a computational investigation is made into the area left open by these theorems.

1.1 Preliminaries

Before starting we should pin down some notation that will be used fairly consistently throughout this thesis.

Let $\mathbb{R}$ denote the real numbers. We will write $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$ for the field of complex numbers. A point $z \in \mathbb{C}$ will be denoted several different ways depending on what the situation deems convenient. One, given in the definition of $\mathbb{C}$, is as a pair of real numbers (x, y). For our purposes another useful representation is by polar coordinates. For this we write

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

where $r \in [0, \infty)$ and $\theta \in \mathbb{R}$. Ocassionally we will express $re^{i\theta}$ as the ordered pair (r, θ). Notice that there are infinitely many values of θ expressing the same z. We will generally be restricting our values for θ to the ranges $[0, 2\pi], [-\pi, \pi], [0, 2\pi)$, or $(-\pi, \pi]$, where the polar representation will be unique, except perhaps at a single point. We will occasionally have the need to express intervals with a centre and radius. In these cases $I(c, \delta)$ will be used to express the interval $[c - \delta, c + \delta]$ or $(c - \delta, c + \delta)$. Whether the interval is open of closed should be clear from the context ro it will be stated explicitly. Due to our multiplicity of notation for points of $\mathbb{C}$, functions on $\mathbb{C}$ will have a similar range of representations. As a rule of thumb, lower-case letters such as f, g, u, v and ϕ will be used to denote real-valued functions on $\mathbb{C}$ whereas upper-case letters like F, G, Φ will respresent complex-valued functions of a complex variable. We can express such an F by

$$F(z) = F(x, y) = u(x, y) + iv(x, y)$$

or

$$F(re^{i\theta}) = F(r, \theta) = u(r, \theta) + iv(r, \theta)$$

where u and v are real-valued. We say $\mathcal{D} \subseteq \mathbb{C}$ is a *domain* in case $\mathcal{D}$ is an open, connected set. The closure of $\mathcal{D}$ will be written as $\overline{\mathcal{D}}$ and the boundary of $\mathcal{D}$, $\overline{\mathcal{D}} \setminus \mathcal{D}$, as $\partial\mathcal{D}$. We now look at some domains we will encounter frequently. Let $B(z_0, r)$ denote the open disk $\{z : |z - z_0| < r\}$. We will commonly use Δ as a shorthand for $B(z_0, r)$. The boundary of these domains are circles of radius r which we will denote by C_r.

Let K be a closed subset of $[0, 1]$. A *radial slit* is a set of the form $zK = \{\lambda z : \lambda \in K\}$, and an *arrangement of slits* is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $0 \leq \alpha_1 \leq\leq \cdots \leq \alpha_n \leq 2\pi$. For a fixed n we will say that $\tilde{\alpha}$ is an *evenly spaced* arrangement of slits if

$$\tilde{\alpha}_j = \frac{(j-1)\pi}{n}$$

for $1 \leq j \leq n$. We call Ω_α a *radially slit disk* if $\Omega_\alpha = \Delta \setminus S_\alpha$ where

$$S_\alpha = \bigcup_{j=1}^{n} e^{i\alpha_j}.$$

Throughout this thesis, Ω_α and S_α will be shortened to Ω and S when it is clear what the arrangement is. We will also use $\tilde{\Omega}$ and $\tilde{S}$ to denote $\Omega_{\tilde{\alpha}}$ and $S_{\tilde{\alpha}}$ repsectively.

If a function f is continuous on some $E \subseteq \mathbb{C}$ we write $f \in C(E)$. Furthermore, if f has continuous kth-order derivatives we say $f \in C^k(E)$. The derivative with respect to the complex variable z is denoted by a prime,

$$f'(z) = \frac{d}{dz} f(z).$$

When considering a function of a complex variable as a function on $\mathbb{R}^2$ we will denote the partial derivatives of a functions with respect to a variable by the function subscripted by that variable. For example,

$$f_x(x, y) = \frac{\partial}{\partial x} f(x, y) \quad , \quad f_\theta(re^{i\theta}) = \frac{\partial}{\partial \theta} f(re^{i\theta}).$$

Before moving on to the next sections, we define a class of functions that will be used throughout.

Definition 1.1.1. Let $\mathcal{D}$ be a domain. A function $F : \mathcal{D} \to \mathbb{C}$ is said to be *analytic on* $\mathcal{D}$ if F has a derivative at every $z \in \mathcal{D}$.

Proposition 1.1.2. *An analytic function $F = u + iv$ satisfies the* Cauchy-Riemann *equations,*

$$u_x = v_y \quad , \quad u_y = -v_x \tag{1.1}$$

A proof of this can be found in [2].

Any other notation used, but not explicitly stated here, can generally be found in one of [2], [10], or [12].

Using this section as a general point of reference, statements of the two major theorems will be given here.

Suppose Ω and $\tilde{\Omega}$ are radially slit domains, both with n slits all formed from the set K. Let $v(z) = \omega(z, \partial\Omega, \Omega)$, the *harmonic measure* of $\partial\Omega$ with respect to the domain Ω. To clarify, $v(z)$ is the bounded, harmonic function on Ω_α satisfying the boundary conditions $v(z) = 1$ for $z \in \partial\Omega$ and $v(z) = 0$ on S. Similarly, define $u(z) = \omega(z, \partial\tilde{\Omega}, \tilde{\Omega})$. A precise formulation of harmonic measure is deferred until the next section. We can now state the theorems.

Theorem 1.1.3 (Dubinin's Theorem)**.** *Suppose the domains Ω and $\tilde{\Omega}$ are simply connected, that is, $K = [a, 1]$ for some $0 < a < 1$. Then the functions u and v satisfy*

$$u(0) \leq v(0)$$

with strict inequality unless Ω can be obtained from $\tilde{\Omega}$ by a rotation about the origin.

Theorem 1.1.4 (Baernstein's Theorem)**.** *Suppose $n \leq 3$ and K is an arbitrary closed subset of $[0, 1]$. Let $\Phi : [0, 1] \to \mathbb{R}$ be any increasing, convex function. Then for each $r \in (0, 1)$,*

$$\int_{-\pi}^{\pi} \Phi(u(re^{i\theta}))\, d\theta \leq \int_{-\pi}^{\pi} \Phi(v(re^{i\theta}))\, d\theta.$$

1.2 Harmonic Measure

In this section we introduce the concepts needed to define precisely what was meant by "harmonic measure" in the statement of Theorem 1.1.3 and Theorem 1.1.4.

Definition 1.2.1. A function $u : \mathcal{D} \to \mathbb{R}$ is said to satisfy the mean value property on $\mathcal{D}$ if, whenever $\overline{B(z_0, r)} \subseteq \mathcal{D}$ we have

$$u(z_0) = L(u; z_0, r) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z_0 + re^{i\theta})\, d\theta. \tag{1.2}$$

This property states that for every point in $\mathcal{D}$, the value of the function, u, at this point is the average of the values of the function at the points immediately surrounding it. We will use this property to characterise what we mean by harmonic functions.

Definition 1.2.2. A function $u : \mathcal{D} \to \mathbb{R}$, continuous on $\mathcal{D}$ is said to be *harmonic on $\mathcal{D}$* if it satisfies the mean value property on $\mathcal{D}$.

Another fairly common way of defining harmonic functions is to require u to have continuous second-order derivatives and that these derivatives satisfy *Laplace's equation*,

$$\Delta u \stackrel{\text{def}}{=} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1.3}$$

The symbol Δ also appears in the literature as ∇^2. With either notation it is called the laplacian. When using Δ, the context should make it clear whether we are talking about the laplacian or the open unit disk. The next proposition tells of an important realtion between harmonic and analytic functions.

Proposition 1.2.3. *The real and imaginary parts of an analytic function on a domain $\mathcal{D}$ are harmonic functions.*

Proof. The proof is by evaluating the laplacian of the real and imaginary parts and using the Cauchy-Riemann equations. The details can be found in [2]. □

Requiring $u \in C^2(\mathcal{D})$ turns out to be too restrictive for our purposes. It can be shown (see [2]) that these definitions are equivalent and so any function harmonic in the sense of Definition 1.2.2 has derivatives of all orders. The definition using the mean value property gives a better "feel" for what a harmonic function looks like, as we shall see. The dependency of a harmonic function's value on the value at surrounding points makes it easy to define with relatively little information. Before we can see this, we need to know about a property of harmonic functions known as the *maximum principle.*

Proposition 1.2.4. *Suppose u is a harmonic function on $\mathcal{D}$ that is continuous on $\overline{\mathcal{D}}$. Then u takes its maximum and minimum on the boundary of $\mathcal{D}$.*

The proof of this stems from the mean value property of harmonic functions as can be seen in [2]. We now look at how the boundary values of a harmonic function can determine the function over its entire domain of defintion. The following example is illustrative of what to expect.

Example 1.2.5. Suppose that a function $u : \overline{\Delta} \to \mathbb{R}$ is required to be harmonic on Δ, continuous on $\overline{\Delta}$ and $u(\zeta) = c$, a constant, for all $\zeta \in \partial\Delta$. The constant function $u(z) = c$ satisfies all these requirements, and in fact is the only such function. The fact u is a solution to the given problem is trivial. To show uniqueness we will prove a stronger result. Namely, if u is a harmonic function on Δ, continuous on $\overline{\Delta}$ and equal to some bounded, continuous function f on $\partial\Delta$, then u is unique. Assume v is another function satisfying these conditions. Then the difference, $w(z) = v(z) - u(z), z \in \Delta$ must also be harmonic as L is easily seen to be a linear operator, so for any $B(z_0, r) \subseteq \Delta$,

$$\begin{aligned} w(z_0) &= v(z_0) - u(z_0) \\ &= L(v; z_0, r) - L(u; z_0, r) \\ &= L(v - u; z_0, r) \\ &= L(w, z_0, r). \end{aligned}$$

This difference is also continuous on Δ and must be zero on $\partial\Delta$. By the maximum principle we know $w(z) \leq 0$ on $\overline{\mathcal{D}}$. As $-w$ is also harmonic on $\mathcal{D}$ by the linearity of L, applying the maximum principle again gives us $w(z) \geq 0$ on $\overline{\mathcal{D}}$. Hence $w(z) = v(z) - u(z) = 0$ for all $z \in \overline{\mathcal{D}}$ and this gives us uniqueness.

Provided that we can find a function u that satisfies these conditions we can then apply the above argument to ensure that u is unique. The problem of finding such a function given a domain and boundary conditions is known as the *Dirichlet Problem*. We now give a precise formulation of this problem.

Definition 1.2.6. Let $\mathcal{D}$ be a bounded domain. Let $f : \partial\mathcal{D} \to \mathbb{R}$ be a bounded and continuous function. A function $u : \mathcal{D} \to \mathbb{R}$ is said to solve the Dirichlet problem on $\mathcal{D}$ with boundary conditions f if

1. u is harmonic on $\mathcal{D}$ and
2. $\lim\limits_{z\to\zeta} u(z) = f(\zeta)$ for all $\zeta \in \partial\mathcal{D}$.

The second condition can be fulfilled by requiring u to be continuous on $\overline{\mathcal{D}}$ and $u(\zeta) = f(\zeta)$ for all $\zeta \in \partial\mathcal{D}$.

Dirichlet problems have their roots in trying to describe the equilibrium state of some physical system. The boundary conditions given by f can be used to represent a fixed distribution of charge or temperature on the boundary of some conductive surface. The function, u, then gives a description of the electrical or thermal potential (in fact, the term “harmonic” is synonymous with “potential” when talking of these functions). It is important to notice that this potential u describes is when the system has reached a state of equilibrium. Taking a thermal system as an example, imagine “switching on” the heat at the boundary. Initially, waves of heat will flow through the domain from higher to lower areas of temperature. Eventually the system will spread out the heat evenly over the domain. This “spreading out” idea is precisely what is being described in the mean value property of harmonic functions in Definition 1.2.1. Thinking of Dirichlet problems in the physical sense tends to make understanding properties of the solution u somewhat easier.

The following example shows that the Dirichlet problem does not always have a solution, it also gives a taste for the type of arguments that will permeate this thesis. Not always having an exact solution is not a huge setback as we will subsequently give a method of finding functions that are “almost” solutions to a given Dirichlet problem.

Example 1.2.7. Let $\mathcal{D} = \Delta \setminus \{0\}$ and define f by $f(0) = -1, f(\zeta) = 0$ for $\zeta \in \partial\Delta$. f is then a continuous function on $\partial\mathcal{D}$, however there is no function u defined on $\mathcal{D}$ that solves the Dirichlet problem for this domain and boundary conditions. Suppose we have a solution, u. By Proposition 1.2.3 we see that the real-valued function $\varepsilon \log|z|$ is harmonic on $\mathcal{D}$ as it is the real part of the analytic function $\varepsilon \log z$. Therefore the function

$$v(z) = u(z) - \varepsilon \log|z|$$

is harmonic on $\mathcal{D}$ and zero on $\{z : |z| = 1\}$ for all ε. Also, as $-\log|z|$ is unbounded and positive as $|z| \to 0$, we see the boundary value at 0 is $+\infty$. Hence, by the maximum principle applied to v we get, for each $z \in \mathcal{D}$

$$u(z) \geq \varepsilon \log|z|$$

for all $\varepsilon > 0$. Thus u is identically zero on all of $\mathcal{D}$ contradicting our assumption that

$$\lim_{z \to 0} u(z) = -1.$$

It seems that in this example the point at the middle of the disk was in a sense "too small" to be "seen" by the function u. What is about to be described is a method, due to O. Perron, of finding solutions to Dirichlet problems that may not satisfy condition 2 of Definition 1.2.6 on a "small" set of points. The explanation of the Perron process given here is based on [12]. We will need the notion of *subharmonic functions*.

Definition 1.2.8. An upper semi-continuous function $u : \mathcal{D} \to \mathbb{R}$ is said to be *subharmonic on $\mathcal{D}$* if, whenever $\overline{B(z_0, r)} \subseteq \mathcal{D}$ we have

$$u(z_0) \leq L(u; z_0, r). \tag{1.4}$$

By upper semi-continuous (u.s.c.) we mean the sets $\{z \in \mathcal{D} : u(z) < a\}$ are open for all $a \in (-\infty, \infty)$. We ask this of u so the integral $L(u; z_0, r)$ is bounded above since u u.s.c. implies u is bounded from above. To complete the trinity we also give the definition of a *superharmonic* function here as we will need it in later sections.

Definition 1.2.9. A lower semi-continuous function $u : \mathcal{D} \to \mathbb{R}$ is called *superharmonic on $\mathcal{D}$* if

$$u(z_0) \geq L(u; z_0, r) \tag{1.5}$$

whenever $\overline{B(z_0, r)} \subseteq \mathcal{D}$.

A function is called lower semi-continuous if its negative is upper semi-continuous. In fact, we could have defined a superharmonic function as one whose additive inverse is subharmonic. The advantage of stating each definition explicitly is that it allows us to easily see how each type of function relates to its mean. We will now quickly look at some of the properties of these functions and how they relate to one another.

It is clear that a harmonic function is both superharmonic and subharmonic, and conversely, if a given function is both superharmonic and subharmonic it is necessarily harmonic. Also each class is closed under addition and scalar multiplication by a non-negative number. If a superharmonic function is multiplied by a negative real, the resulting function is subharmonic, and vice-versa. Suppose that u, v and w, defined on a domain $\mathcal{D}$, are respectively subharmonic, harmonic and superharmonic functions. If the boundary values of u, v and w are the same then it can be shown that for all $z \in \mathcal{D}$

$$u(z) \leq v(z) \leq w(z).$$

If, instead, we ask that $u, v, w \in C^2(\mathcal{D})$ it can be shown that

$$\Delta u \geq 0 \quad , \quad \Delta v = 0 \quad , \quad \Delta w \leq 0.$$

One final thing to note about super and subharmonic functions is that they also satisfy a something similar to the maximum principle. Recall that the values of harmonic functions are bounded strictly inbetween it boundary values. It turns out that superharmonic functions are strictly greater than their boundary values and subharmonic functions strictly less than their boundary values. These facts are commonly called the *minimum principle for superharmonic functions* and the *maximum principle for subharmonic functions* respectively.

We will not encounter superharmonic functions again until Section 3.2, where their properties are used in the proof of Theorem 1.1.4. Concentrating on subharmonic functions, we turn back to the problem at hand — solving the Dirichlet problem.

For a given Dirichlet problem on $\mathcal{D}$ with boundary conditions $f \in C(\partial\mathcal{D})$ we will say a subharmonic function u is *bounded by f on $\partial\mathcal{D}$* if it is bounded above and for all $\zeta \in \partial\mathcal{D}$

$$\limsup_{z \to \zeta} u(z) \leq f(\zeta).$$

We then let $\mathcal{S}_f$ be the set of all functions bounded by f on $\partial\mathcal{D}$ and denote by u_f the pointwise defined function

$$u_f(z) = \sup\{u(z) : u \in \mathcal{S}_f\} \qquad (z \in \mathcal{D}). \tag{1.6}$$

Somewhat surprisingly this function is harmonic, bounded, and moreover, the solution of our Dirichlet problem, except perhaps at a small set of points. The proof that u_f is a bounded, harmonic function can be found in [2] or [12]. To see how well u_f solves the given Dirichlet problem we call any point ζ_0 on $\partial\mathcal{D}$ *regular* if, for every bounded, continuous function f,

$$\lim_{z \to \zeta_0} u_f(z) = f(\zeta_0). \tag{1.7}$$

Naturally, if a point is not regular it is deemed to be *irregular*. Notice that to be a regular point the condition in (1.7) must be satisfied for *every* continuous, bounded function f, not just the function dictating the boundary conditions. Intuitively then, it seems that the property of having regular points is one more likely due to the geometry of the domain

$\mathcal{D}$ than the boundary conditions imposed by some function f defined on $\partial\mathcal{D}$. In fact, in [2] and [12] geometric conditions are given, which, if satisfied by a domain, ensure the boundary of that domain contains only regular points. On such a domain (called a *regular domain*) we see that u_f is a solution for given boundary conditions f. Even when the domain does contain irregular points they form a subset of $\partial\mathcal{D}$ with what is known as *capacity zero*. For a precise definition of capacity see [12] or [1]. Earlier, we likened the solution of a Dirichlet problem to an electrostatic potential on a domain with a charge distributed over it boundary. With this analogy in mind, capacity is the amount of charge a subset of the boundary can hold. Slightly more formally, it is the maximum of all regular probability measures on Borel sets of $\partial\mathcal{D}$. This means if a set has capacity zero it also has measure zero for every one of these measures. This said, a set of capacity zero is a very small set indeed. An example of a capacity zero set is the single point boundary of Example 1.2.5.

We have now seen a method of producing functions u_f on a domain $\mathcal{D}$ from the information given about how this function should behave near the boundary of $\mathcal{D}$. We saw in the argument given in Example 1.2.5 that each $f \in C(\partial\mathcal{D})$ must define u_f uniquely, so it is possible to talk of a map from $C(\partial\mathcal{D})$ to the bounded, harmonic functions on $\mathcal{D}$. Here is the major theorem regarding such a map.

Theorem 1.2.10. *Let $\mathcal{D}$ a domain in $\mathbb{C}$ such that the capacity of $\partial\mathcal{D}$ is strictly positive. Then for each $z_0 \in \mathcal{D}$, the map $\Lambda_{z_0} : C(\partial\mathcal{D}) \to \mathbb{R}$, given by*

$$\Lambda_{z_0}(f) = u_f(z_0) \tag{1.8}$$

is a positive linear functional on $C(\partial\mathcal{D})$ of norm one.

By *linear* we mean for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in C(\partial\mathcal{D})$ we have

$$\Lambda_{z_0}(\lambda f + \mu g) = \lambda\Lambda_{z_0}(f) + \mu\Lambda_{z_0}(g).$$

Positive means for all $f \in C(\partial\mathcal{D})$ with $f \geq 0$,

$$\Lambda_{z_0}(f) \geq 0,$$

and the norm being used is

$$\|\Lambda_{z_0}\| = \sup_{f \in C(\partial\mathcal{D})} |\Lambda_{z_0}(f)|.$$

The proof of this theorem is quite technical and can be found in [12]. We are now in a position to give a definition of the title of this section: harmonic measure.

The reason Theorem 1.2.10 is so important is that it tells us that the map Λ_{z_0} satisfies the conditions of the Riesz Representation Theorem (see, for example [17]). This means there exists a unique, regular, Borel measure ω_{z_0} on $\partial\mathcal{D}$ such that

$$\Lambda_{z_0}(f) = \int_{\partial\mathcal{D}} f \, d\omega_{z_0}. \tag{1.9}$$

Definition 1.2.11. The measure ω_{z_0} is called the *harmonic measure on $\partial\mathcal{D}$ for the point z_0 with respect to $\mathcal{D}$*. If E is a Borel subset of $\partial\mathcal{D}$ we will write

$$\omega(z_0, E, \mathcal{D}) \tag{1.10}$$

for the *harmonic measure of E at z_0 with respect to $\mathcal{D}$*.

This is what was meant by the harmonic measure ω in Section 1.1. Although measures are usually considered as functions on sets we will be mainly concerned with a fixed $E \subset \partial\mathcal{D}$ and a varying z_0 or $\mathcal{D}$. Before we get too far ahead of ourselves, it should be mentioned that the radially slit disks Ω_α are domains whose boundary have strictly positive capacity so we can, in fact, induce a harmonic measure on $\partial\Omega_\alpha$ using Theorem 1.2.10. When considering these radially slit disk we will be fixing $E = \partial\Delta$ and looking at what happens to $\omega(z, E, \Omega_\alpha)$ as a function of z as we vary the domain by changing α. We now state some important properties of $\omega(z, E, \mathcal{D})$.

Proposition 1.2.12. *Let $E \subseteq \partial\mathcal{D}$ be Borel. Then,*

1. *$\omega(z, E, \mathcal{D})$ is a harmonic function of z on $\mathcal{D}$. Furthermore,*

   $$\omega(z, E, \mathcal{D}) = u_{\chi_E}$$

   where u_{χ_E} is the function obtained by the Perron process on the domain $\mathcal{D}$ with boundary conditions given by

   $$\chi_E(\zeta) = \begin{cases} 1 & \zeta \in E, \\ 0 & \zeta \notin E. \end{cases}$$

2. *If E is relatively open,*

   $$\lim_{z\to\zeta} \omega(z, E, \mathcal{D}) = 1$$

   for all $\zeta \in E$ except perhaps a set of capacity zero.

3. *If E is compact,*

   $$\lim_{z\to\zeta} \omega(z, E, \mathcal{D}) = 0$$

   for all $\zeta \in \partial\mathcal{D} \setminus E$ except perhaps a set of capacity zero.

This proposition is stated as several theorems with proofs in [12].

Figure 1.1 was generated using the `PLTMG` package discussed in Section 4.2. It is a computer approximation of the harmonic measure of the slits of a three slit domain. It is presented here to help us visualise this type of function and how it it depends on its domain.

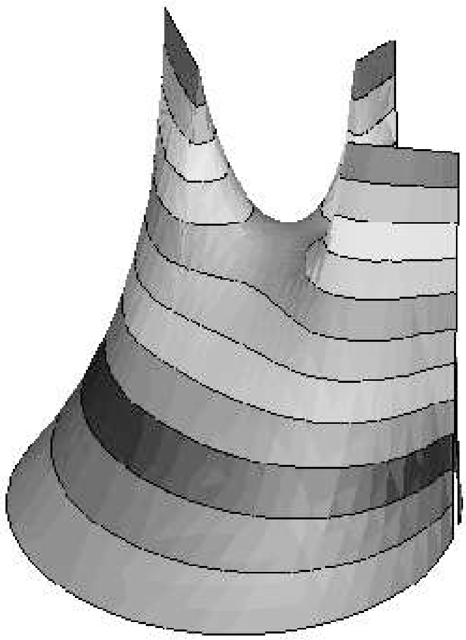

Figure 1.1: Harmonic Measure

It is worth pointing out that this harmonic measure is "upside-down" compared to those in the statement of Theorem 1.1.3. The function shown here takes boundary values 1 on S and 0 on $\partial\Delta$, while in Dubinin's theorem this is the other way round. It turns out that it is slightly easier to prove Dubinin's theorem using functions like those in the figure. To justify being able to do this we will need the following proposition.

Proposition 1.2.13. *Let $\mathcal{D}$ be a domain and suppose $S, T \subset \partial\mathcal{D}$ are closed and satisfy*

$$S = \partial\mathcal{D} \setminus T \cup Z$$

where Z is a subset of $\partial\mathcal{D}$ of capacity zero. Let $\omega(z, T, \mathcal{D})$ be the harmonic measure of the set T with respect to the domain $\mathcal{D}$. Then $1 - \omega(z, T, \mathcal{D}) = \omega(z, S, \mathcal{D})$, the harmonic measure of S with respect to $\mathcal{D}$.

Proof. It is clear that $\Delta(1 - \omega(z, T, \mathcal{D})) = 0$ and furthermore the boundary values of $1 - \omega(z, T, \mathcal{D})$ are given by $\chi_S(\zeta)$ as we can ignore the set of capacity zero by Proposition 1.2.12. Therefore $1 - \omega(z, T, \mathcal{D}) = \omega(z, S, \mathcal{D})$. □

Also needed for our proofs is a way of transforming domains and functions on domains whilst preserving crucial properties of both. This is focus of the next section.

1.3 Some Conformal Map Results

This section hopes to give a brief overview of conformal maps and some of their properties that will be useful in proving Dubinin's Theorem. Most of the elementary theory given here is based on [2] and [11]. The material discussing conformal mapping of multiply connected domains is from [14].

As mentioned in the previous section (Proposition 1.2.3), the real and imaginary parts of an analytic function on a domain are harmonic in that domain. Furthermore, if the domain, $\mathcal{D}$, is simply connected we can construct explicitly a *harmonic conjugate*, v, for a given harmonic function, u, on this domain. By this we mean u and v satisfy the Cauchy-Riemann equations (1.1). If this is possible the function $u + iv$ is then clearly analytic on $\mathcal{D}$. Proof of these statements can be found in [11]. We need them to give a simple proof of what will be one of the essential ingredients in the proof of Dubinin's theorem.

Proposition 1.3.1. *Suppose f is a harmonic function on $\mathcal{D}$, and $\Phi : \mathcal{D}' \to \mathcal{D}$ is analytic. Then the function g defined on $\mathcal{D}'$ by*

$$g(z) = f(\Phi(z)) \qquad (z \in \mathcal{D})$$

is harmonic on $\mathcal{D}'$.

Proof. As $\mathcal{D}$ is a domain it is open and therefore about any point $z_0 \in \mathcal{D}$ we can find a ball $B(z_0, r)$ sitting entirely in $\mathcal{D}$. As harmonicity is a local property f is harmonic in this ball which is obviously a simply connected domain. We therefore can find an analytic function F on $B(z_0, r)$ for which u is the real part of F. The real part of the function

$$\hat{F}(z) = F(\Phi(z)) \qquad (z \in B(z_0, r))$$

is $f(\Phi(z)) = g(z)$ and must be harmonic in $B(z_0, r)$ as $\hat{F}$ is the composition of two analytic functions and therefore analytic. This argument holds for any such z_0 in $\mathcal{D}$ and so g must be harmonic in $\mathcal{D}$. □

We digress for a moment and look at a special class of analytic functions called *conformal maps*. These maps have pleasing geometric properties and, as we will see, are useful when trying to find more explicit solutions to Dirichlet problems than the Perron process can supply.

Definition 1.3.2. A *conformal* map F, on a domain $\mathcal{D}$, is an analytic function on $\mathcal{D}$ with non-vanishing derivative. That is,

$$F'(z) \neq 0$$

for all $z \in \mathcal{D}$.

What follows is a few examples of conformal maps that will be used elsewhere in this thesis.

Example 1.3.3. We will denote an open rectangle in the complex plane by

$$R = (a, b) \times (c, d) \stackrel{\text{def}}{=} \{x + iy \in \mathbb{C} : a < x < b, c < y < d\}.$$

For this example we will require $R = (a, 0) \times (0, \theta)$ where $a < 0$ and $0 < \theta < 2\pi$. Let $F : \mathbb{C} \to \mathbb{C}$ be the map $z \mapsto e^z$. By taking derivatives it is easy to check that F is a conformal map on R. Furthermore,

$$\begin{aligned} F(R) &= \{F(x + iy) : a < x < 0, 0 < y < \theta\} \\ &= \{e^x e^{iy} : a < x < 0, 0 < y < \theta\} \\ &= \mathcal{D}_{r_0}(r_0, 0) \end{aligned}$$

where, $r_0 = e^a$ and

$$\mathcal{D}_r(r, \theta_0) \stackrel{\text{def}}{=} \{\rho e^{i\theta} : r < \rho < 1, \theta_0 < \theta < \theta_1\}.$$

We will often write $\mathcal{D}(\theta_0, \theta_1)$ for $\mathcal{D}_0(0, \theta_0)$ and call these type of domains *sectors*. We see then that $z \mapsto e^z$ allows us to take rectangles to sectors, and the inverse of F, $z \mapsto \log z$, takes sectors to rectangles. Here the logarithm can be chosen so its branch cut lies outside the sector.

Example 1.3.4. Let $\mathcal{D}(0, \theta)$ be as described above. For each $t > 0$, a real number, define

$$P_t(z) = z^t \stackrel{\text{def}}{=} e^{t \log z}.$$

Then,

$$\begin{aligned} P_t(\mathcal{D}(0, \theta_1)) &= \{P_t(re^{i\theta}) : 0 < r < 1, 0 < \theta < \theta_1\} \\ &= \{r^t e^{it\theta} : 0 < r < 1, 0 < \theta < \theta_1\} \\ &= \{\rho e^{i\varphi} : 0 < \rho < 1, 0 < \varphi < t\theta_1\} \\ &= \mathcal{D}(0, t\theta_1). \end{aligned}$$

This map allows us to "stretch" a sector around the origin. An inspection of P_t reveals it is nothing more than a composition of the two maps mentioned in the previous example along with a multiplication by t. We can think of this map as first taking the sector $\mathcal{D}(0, \theta_1)$ and mapping it onto the semi-infinite rectangle $R_1 = (-\infty, 0) \times (0, \theta_1)$ using $\log z$. The map $z \mapsto tz$ takes this rectangle conformally onto the rectangle $R_2 = (-\infty, 0) \times (0, t\theta_1)$. Using $z \mapsto e^z$ we take this rectangle onto the sector $\mathcal{D}(0, t\theta)$. This shows that P_t is nothing more than a composition of three conformal maps, hence conformal. Alternately, if we take the derivative of this map with respect to z we obtain tz^{t-1} which vanishes only at $z = 0$. However $0 \notin \mathcal{D}(0, \theta_1)$ so P_t is again conformal on $\mathcal{D}(0, \theta_1)$.

We define now a map from the upper half disk to the lower half plane called the *Joukowski map*.

Example 1.3.5. Let $\Delta^+ = \{x+iy \in \Delta : y > 0\}$ and $H^- = \{x+iy \in \mathbb{C} : y < 0\}$. Define $W : \Delta^+ \to H^-$ by

$$W(z) = \frac{z + z^{-1}}{2}.$$

A quick calculation show that

$$W'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

and so W is conformal every where in $\mathbb{C}$ except 0 where it isn't analytic, and ± 1 where the derivative vanishes. Hence W is conformal in Δ^+.

Notice that if z is in $(0,1)$ or (-1,0) then $W(z)$ is in $(1,\infty)$ or $(-\infty,0)$ respectively. Also, if $z = x+iy$ and $|z| = 1$ then

$$\begin{aligned} W(z) &= \frac{1}{2}\left(z + \frac{\bar{z}}{|z|}\right) \\ &= \frac{1}{2}((x+iy) + (x-iy)) = x, \end{aligned}$$

so $W(\{z \in \mathbb{C} : |z| = 1\}) = [-1, 1]$. The boundary of Δ^+ therefore gets mapped onto all of the real line. As $\frac{i}{2}$ is taken to $\frac{-3i}{2}$, we see W conformally maps Δ^+ to H^-.

We will now use these maps to explicitly calculate the harmonic measure at zero of a simply connected domain with evenly spaced slits.

Suppose $\tilde{\Omega}$ is a radially slit domain with n evenly spaced slits, $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$ constructed from the interval $K = [a,1], 0 \le a < 1$. Let $\mathcal{S} = \mathcal{D}(0, \frac{2\pi}{n})$. Our aim here is to conformally map this sector onto itself and move the subset of the boundary lying on $[a,1]$ to the circular part of the boundary.

Once this is done we will use the *Schwarz Reflection Principle* to extend this map to one that takes $\tilde{\Omega}$ onto Δ and the slits of $\tilde{\Omega}$ onto $\partial\Delta$. The philosophy behind the reflection principle is that symmetry in a domain can induce symmetry in an analytic function defined on that domain. More precisely, suppose $\mathcal{D}$ is a domain symmetric about some line l. Then if F is an analytic function defined on one side of $\mathcal{D}$, we can analytically extend F over the line l to the other side of $\mathcal{D}$. This extended function will have aspects of symmetry reflecting that of the domain. To properly speak of analytic extensions and the general Schwarz reflection principle would lead us too far astray. See [2] or [14] for a precise statement of this principle. We will want to use this to reflect $\mathcal{S}$ in the line $\{z : \arg z = \frac{2\pi}{n}\}$ and extend the conformal map to this domain. Subsequent reflections in $\arg z = \frac{2j\pi}{n}$ for $j = 2, \ldots, n$ will extend the map to all of $\tilde{\Omega}$. As this extended map will be analytic it will also be conformal, since the derivative of the map on $\mathcal{S}$ doesn't vanish.

Let $t = \frac{n}{2}$. The map P_t from Example 1.3.4 for this value of t will then map $\mathcal{S} = \mathcal{D}(0, \frac{2\pi}{n})$ to the upper half disk $\Delta^+ \equiv \mathcal{D}(0, \pi)$. The slit $[a,1]$ on the boundary of $\mathcal{S}$ will be taken to the interval $[b,1]$ where $b = a^{\frac{n}{2}}$. The other slit lying on the boundary of $\mathcal{S}$, $\{re^{i\frac{2\pi}{n}} : a \leq \theta \leq 1\}$, will get taken to $[-1,-b]$. Applying the Joukowski map W of Example 1.3.5 to Δ^+ will map $[b,1]$ on the boundary of Δ^+ to $[1,c]$ where $c = W(b) > 1$. Similarly, $[-1,-b]$ gets mapped to $[-c,-1]$. We now dilate the entire complex plane by a factor of $\frac{1}{c}$ with the map $z \mapsto \frac{1}{c}z$, which is clearly conformal everywhere. This has the effect of taking the intervals $[-c,-1],[1,c]$ on the boundary of H^- to the intervals $[-1,-d],[d,1] \subset \partial H^-$, where $d = \frac{1}{c} < 1$. The map $W^{-1} : H^- \to \Delta^+$ can be checked to be conformal. Since W takes $\{z \in \mathbb{C} : |z| = 1\}$ onto $[-1,1]$, W^{-1} will take $[d,1]$ onto some interval on the unit circle. Let $e^{i\beta} = W^{-1}(d)$. Then,

$$d = W(e^{i\beta}) = \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right) = \cos\beta.$$

Hence,

$$\cos\beta = \frac{1}{c} = \frac{1}{W(b)} = \frac{2b}{b^2+1} = \frac{2a^{\frac{n}{2}}}{a^n+1}$$

and therefore,

$$\beta = \cos^{-1}\left(\frac{2a^{\frac{n}{2}}}{a^n+1}\right). \tag{1.11}$$

After conformally mapping the upper half disk back onto $\mathcal{S}$ via $z \mapsto z^{\frac{2}{n}}$, we see that the slit $[a,1]$ gets mapped to the set

$$\{z : |z| = 1, 0 \leq \arg z \leq \frac{2}{n}\beta\}.$$

Also, the slit $\{re^{i\frac{2\pi}{n}} : a \leq \theta \leq 1\}$, will be taken to

$$\{z : |z| = 1, \frac{2(\pi-\beta)}{n} \leq \arg z \leq \frac{2\pi}{n}\}.$$

We will denote the composition of all these maps $\tilde{F}$. We can now extend $\tilde{F}$ to a conformal map $\tilde{F} : \tilde{\Omega} \to \Delta$ by the reflection principle as discussed on page 16. As $\tilde{F}$ is conformal, it is therefore analytic, as is $\tilde{F}^{-1}$. This means by Proposition 1.3.1,

$$f(z) = \omega(\tilde{F}^{-1}(z), \tilde{S}, \tilde{\Omega})$$

is harmonic on $\partial\Delta$, where $\tilde{S}$ denotes the slits of $\tilde{\Omega}$. If we continue f to $\partial\Delta$ we get that

$$f(\zeta) = \chi_{\tilde{F}(\tilde{S})}(\zeta)$$

for $\zeta \in \partial\Delta$. By the mean value property of f,

$$\begin{aligned} f(0) &= \frac{1}{2\pi}\int_{\partial\Delta} f(w)\,dw \\ &= \frac{1}{2\pi}\int_{\partial\Delta} \chi_{\tilde{F}(\tilde{S})}(w)\,dw \\ &= \frac{1}{2\pi}|\tilde{F}(\tilde{S})| \end{aligned} \tag{1.12}$$

where $|\tilde{F}(\tilde{S})|$ is the Lebesgue measure of the image of the slits on $\partial\Delta$. By the construction of $\tilde{F}$ it is clear that the length of the image of one of the slits is given by

$$2|\{z : |z| = 1, 0 \leq \arg z \leq \frac{2}{n}\beta\}| = \frac{4\beta}{n}.$$

The symmetry of $\tilde{F}$ implies the total length of the image of the slits will be n times this. Substituting this value back into (1.12) gives $f(0) = \frac{2\beta}{\pi}$. Since $\tilde{F}^{-1}(0) = 0$, $f(0) = \omega(0, \tilde{S}, \tilde{\Omega})$, we have

$$\omega(0, \tilde{S}, \tilde{\Omega}) = \frac{2}{\pi} \cos^{-1}\left(\frac{2a^{\frac{n}{2}}}{a^n + 1}\right) \tag{1.13}$$

giving us an expression for the harmonic measure at zero of n evenly spaced slits of $\tilde{\Omega}$ in terms of n and the slit length $1 - a$.

With only these few conformal maps and the reflection principle at our disposal it is clear that we can conformally map many varied simply connected domains onto one another. The question arises: Given any two simply connected domains, does there exist a conformal map that takes one domain onto the other? This question was first formulated by Riemann and later successfully proved by Koebe. The result is known as the *Riemann Mapping Theorem.*

Theorem 1.3.6. *Given any simply connected domain $\mathcal{D}$ which is a proper subset of $\mathbb{C}$, and a point $z_0 \in \mathcal{D}$ there exists a conformal map, F, from $\mathcal{D}$ to the open unit disk Δ. This map is unique if we require $F(z_0) = 0$ and $F'(z_0)$ to be a positive real number.*

An immediate consequence of this theorem is that given any two simply connected domains and two fixed points within them, $z_0 \in \mathcal{D}, w_0 \in \mathcal{D}'$, there exists a conformal map taking $\mathcal{D}$ to $\mathcal{D}'$ and z_0 to w_0. The reason for this is we can conformally map both domains onto the unit disk, taking z_0 and w_0 to the origin by the Riemann mapping theorem. By composing the map taking $\mathcal{D}$ to the unit disk with the inverse map from $\mathcal{D}'$ to the unit disk, we obtain a conformal map with the required properties.

The proof of the Riemann mapping theorem is, unfortunately, too long to present and is omitted (see [2]). Using this theorem and the above argument we can assert the existence of a conformal map $F : \Omega \to \Delta$ for each simply connected, radially slit domain Ω. If S denotes the n slits of Ω then $F(S)$ is a subset of $\partial\Delta$ consisting of n disjoint components. Using similar arguments to those for $\tilde{F}$ we can show

$$\omega(0, S, \Omega) = \frac{1}{2\pi}|F(S)|.$$

The inequality bewteen harmonic measure in Dubinin's theorem can be reinterpreted as an inequality between the length of the images of the slits. Namely,

Proposition 1.3.7.

$$\omega(0, \partial\Delta, \Omega) \geq \omega(0, \partial\Delta, \tilde{\Omega}) \quad \textit{if and only if} \quad |F(S)| \leq |\tilde{F}(\tilde{S})|. \tag{1.14}$$

Proof. Using Proposition 1.2.13 we can show

$$\omega(0, \partial\Delta, \Omega) = 1 - \omega(0, S, \Omega) \quad \text{and} \quad \omega(0, \partial\Delta, \tilde{\Omega}) = 1 - \omega(0, \tilde{S}, \Omega).$$

Thus, if the left hand side of (1.14) is true then

$$|F(S)| = \omega(0, S, \Omega) \leq \omega(0, \tilde{S}, \tilde{\Omega}) = |\tilde{F}(\tilde{S})|$$

proving one direction of the equivalence. The other direction is shown by a reverse of this argument. □

Another tool we will need in that proof will an extension of the Riemann mapping theorem to doubly connected domains. The canonical domain in the Reimann mapping theorem was the unit disk since every domain that was simply connected could be conformally mapped onto it. When we move up into higher levels of connectivity we can no longer conformally map domains onto the unit disk as conformal maps are continuous and hence preserve connectedness. Instead, we need different types of canonical domain, with at least one for each order of connectedness. For doubly connected domains a natural choice of canonical domain would seem to be the annulus. In fact, in [14], it is shown that any doubly connected domain can be mapped onto an annulus. It is not the case, however, that any doubly connected domain can be mapped onto an *arbitrary* annulus. Suppose r_1 and r_2 are the inner and outer radii of an annulus that a doubly connected domain $\mathcal{D}$ can be conformally mapped to. Then it can be shown that $\mathcal{D}$ can only be mapped to annuli with inner and outer radii in the ratio $\frac{r_2}{r_1}$. This ratio is called the *modulus of the doubly connected domain* $\mathcal{D}$ and is a conformal invariant. If we ask that $\mathcal{D}$ be conformally mapped to an annulus of outer radius 1, the inner radius r_1 is determined completely by the geometry of $\mathcal{D}$. As we shall see in Section 3.1, this will be an important consideration.

1.4 The Dirichlet Integral

Definition 1.4.1. Let f be a real-valued function on the domain $\mathcal{D}$. We denote by $I_{\mathcal{D}}[f]$, the *Dirichlet integral of* f,

$$I_{\mathcal{D}}[f] = \iint_{\mathcal{D}} \left[(f_x)^2 + (f_y)^2\right] \, dx\, dy. \tag{1.15}$$

The integrand above is occasionally shortened to

$$|\nabla f|^2 = (f_x)^2 + (f_y)^2.$$

Historicaly, Dirichlet attempted to find functions, satisfying certain boundary and smoothness conditions, that minimised his integral. In most cases, the extremal function was the harmonic function solving the Dirichlet problem. Intuitively, the reason for this is that the sum of the squares of the partial derivatives used in (1.15) measures, in a sense, how much the function varies over a domain. We mentioned earlier how a harmonic function can respresent an equilibrium state of a physical system. It would seem natural that this equilibrium function is the one that varies least.

As in [7] will call a function f *piecewise smooth in* $\mathcal{D}$ if it has continuous first-order derivatives at all points in $\mathcal{D}$ except possibly on a finite number of smooth arcs and a finite number of points. The Dirichlet integral exists for these piecewise functions and can be rigorously defined as a limit of integrals over closed subdomains converging to $\mathcal{D}$. A function f is said to be *Lipschitz* on $\mathcal{D}$ f if there exists a constant, C, such that

$$|f(z_1) - f(z_2)| \leq C|z_1 - z_2|$$

whenever $z_1, z_2 \in \mathcal{D}$. A function that is Lipschitz on a bounded domain will have a finite Dirichlet integral since the partial derivatives will be bounded by C. Another property worth mentioning about Lipschitz functions it that if we compose a Lipschitz function with an analytic function the result is Lipschitz (with possibly a different constant). The reason for this is the derivative of an analytic function is analytic, thus its absolute value is bounded on its domain by the maximum modulus principle. This, along with the chain rule, validates the claim.

Definition 1.4.2. We will say that a function is *admissible on* $\mathcal{D}$ if it is piecewise smooth and Lipschitz on $\mathcal{D}$.

Before going into some of the more general theory of the Dirichlet integral we begin with a calculation.

Example 1.4.3. Let $R = (a, b) \times (c, d)$ be a rectangle. Define on R the function

$$\mu(x, y) = \frac{d - y}{d - c} \qquad \text{for} \quad (x, y) \in R.$$

We can easily compute the Dirichlet integral of μ to be

$$I_R[\mu] = \iint_R \left[(0)^2 + (\frac{-1}{d-c})^2 \right] dx\, dy \tag{1.16}$$

$$= (d-c)^{-2} \left[(b-a)(d-c) \right] \tag{1.17}$$

$$= \frac{b-a}{d-c}. \tag{1.18}$$

Notice also that $\mu_{xx} = \mu_{yy} = 0$ and so μ is harmonic in R with boundary values $\mu(x, c) = 0, \mu(x, d) = 1$ for all $x \in (a, b)$. The following proposition tells us that this ramp function has minimal the Dirichlet integral when compared to any other addmissible function on R, with the same boundary conditions.

Proposition 1.4.4. *Using the notation in Example 1.4.3 let*

$$\mathcal{X} = \{f \in DI(R) : f(t,c) = 1, f(t,d) = 0, t \in (a,b)\}.$$

Then, for all $f \in \mathcal{X}$,

$$I_R[\mu] \leq I_R[f].$$

Proof. For any $f \in \mathcal{X}$ we see that

$$\begin{aligned} I_R[f] &= \iint_R \left[(f_x)^2 + (f_y)^2\right] \, dy \, dx \\ &\geq \int_a^b \int_c^d (f_y)^2 \, dy \, dx. \end{aligned} \tag{1.19}$$

Applying Hölder's inequality to the inner integral gives

$$\begin{aligned} \int_c^d f_y^2 \, dy &\geq \left[\int_c^d f_y \, dy\right]^2 \left[\int_c^d 1^2 \, dy\right]^{-1} \\ &= \frac{1}{d-c} \end{aligned}$$

as $[f(x,d) - f(x,c)]^2 = 1$ for $f \in \mathcal{X}$. Substituting this back into 1.19 gives us

$$\begin{aligned} I_R[f] &\geq \int_a^b \frac{1}{d-c} \, dx \\ &= I_R[\mu] \end{aligned}$$

as required. □

We can generalize this result to other domains and boundary conditions and get a similar result — the Dirichlet integral is minimized by the harmonic function that solves the Dirichlet problem. This isn't true for arbitrary domains and boundary conditions, and to state precisely the conditions for which it is true will lead us too far afield. However, for the domains we will be concerned with the following results hold.

Proposition 1.4.5. *Let* f *and* g *be real-valued admissible functions that extend continuously to* $\overline{\mathcal{D}}$. *Suppose* $f = g$ *on* $\partial\mathcal{D}$ *and that* f *is harmonic on* $\mathcal{D}$. *Then,*

$$I_{\mathcal{D}}[f] \leq I_{\mathcal{D}}[g]. \tag{1.20}$$

Proof. Let $\varepsilon = g - f$ and expand $|\nabla g|^2$ to get

$$\begin{aligned} I_{\mathcal{D}}[g] &= I_{\mathcal{D}}[f] + \iint_{\mathcal{D}} [\nabla f \cdot \nabla \varepsilon] \, dx \, dy + I_{\mathcal{D}}[\varepsilon] \\ &= I_{\mathcal{D}}[f] - \iint_{\mathcal{D}} \varepsilon \Delta f \, dx \, dy + \int_{\partial\mathcal{D}} \varepsilon \frac{\partial f}{\partial \mathbf{n}} \, d\zeta + I_{\mathcal{D}}[\varepsilon] \end{aligned} \tag{1.21}$$

This follows by one of Green's identities which can be found in textbooks on partial differential equations such as [18]. The notation $\frac{\partial}{\partial \mathbf{n}}$ is used to represent the normal derivative of f on the boundary of $\mathcal{D}$. The important thing to notice here is that ε is zero on $\partial\mathcal{D}$ and Δf vanishes in $\mathcal{D}$. Hence

$$\begin{aligned} I_{\mathcal{D}}[f] &= I_{\mathcal{D}}[f] + I_{\mathcal{D}}[\varepsilon] \\ &\geq I_{\mathcal{D}}[f] \end{aligned}$$

since the Dirichlet integral is always non-negative. This proves the proposition. □

The next proposition states that if we transform our domain by a conformal map, the Dirichlet integral of a similarly transformed function on the new domain is left unchanged. This property is known as *conformal invariance*.

Proposition 1.4.6. *Let $\Phi : \mathcal{D}' \to \mathcal{D}$ be a conformal map, and f an admissible function on $\mathcal{D}$. Then, $f \circ \Phi$ is admissible and*

$$I_{\mathcal{D}'}[f \circ \Phi] = I_{\mathcal{D}}[f].$$

Proof. It is easy to see that $f \circ \Phi$ is admissible since Φ is conformal and so maps smooth arcs to smooth arcs and points to points while preserving the derivatives. We let $\Phi = \phi + i\psi$ and by use of the Cauchy-Riemann equations see that

$$\begin{aligned} |\nabla(f \circ \Phi)|^2 &= f_x^2(\Phi)\phi_x^2 + f_y^2(\Phi)\psi_x^2 + f_x^2(\Phi)\phi_y^2 + f_y^2(\Phi)\psi_y^2 \\ &= (\phi_x\psi_y - \psi_y\phi_x)(f_x^2(\Phi) + f_y^2(\Phi)) \\ &= J[\Phi]|(\nabla f)(\Phi)|^2, \end{aligned}$$

where $J[\Phi]$ is Jacobian of Φ. Substituting this back into the definition of Dirichlet integral, we obtain,

$$\begin{aligned} I_{\mathcal{D}'}[f \circ \Phi] &= \iint_{\mathcal{D}'} |\nabla(f \circ \Phi)|^2 \, dx \, dy \\ &= \iint_{\mathcal{D}'} |(\nabla f)(\Phi)|^2 J[\Phi] \, dx \, dy \qquad (1.22) \\ &= I_{\mathcal{D}}[f] \end{aligned}$$

$$(1.23)$$

as (1.22) is simply a change of variables from $\mathcal{D}$ to $\mathcal{D}'$. □

We can now use this result to find functions that minimise the Dirichlet integral for given boundary conditions on more complicated domains.

Example 1.4.7. Recall the conformal map F, of Example 1.3.3 that mapped a rectangular domain onto a sector of an anulus. Suppose $f \in \mathcal{X}$ as in Proposition 1.4.4 where the rectangle, R, is $(-a, 0) \times (0, \theta_1), a > 0$. Then, $g = f \circ F^{-1} \in \mathcal{Y}$, where

$$\mathcal{Y} \stackrel{\text{def}}{=} \{g \in C^1(\mathcal{D}_\delta(\delta, 0)) : g(r) = 1, g(re^{i\theta_1}) = 0, r \in (\delta, 1)\},$$

and $\delta = e^{-a} < 1$. By the above proposition we know that the Dirichlet integral of all such g will be equal to the Dirichlet integral for the corresponding $f \in \mathcal{X}$. By Proposition 1.4.4 we know $\mu \in \mathcal{X}$ has the smallest Dirichlet integral, hence,

$$\nu \stackrel{\text{def}}{=} \mu \circ (F^{-1})$$

must have the minimal Dirichlet integral for functions in $\mathcal{Y}$. Furthermore,

$$I_{\mathcal{D}_\delta(\delta,0)}[\nu] = I_R[\mu] = \frac{a}{\theta_1}.$$

We now conclude this section with calculation of a Dirichlet integral for a function that will be required in the proof of Dubinin's theorem.

Example 1.4.8. Let $\mathcal{A}(r_1, r_2)$ denote an open annulus of outer radius r_2 and inner radius r_1. That is,

$$\mathcal{A}(r_1, r_2) \stackrel{\text{def}}{=} \{z \in \mathbb{C} : r_1 < |z| < r_2\}.$$

Let u be the solution of the Dirichlet problem on $\mathcal{A}(r_1, r_2)$ with $u(r_1e^{i\theta}) = 1, u(r_2e^{i\theta}) = 0, \theta \in [-\pi, \pi]$ as boundary conditions. Then u is of the form

$$u(re^{i\theta}) = A \log r + B$$

for some real constants A and B. To see this we will require the *polar form of the laplacian*, namely,

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Applying this to u we get

$$\begin{aligned}(\Delta u)(re^{i\theta}) &= (A \log r + B)_{rr} + \frac{1}{r}(A \log r + B)_r + \frac{1}{r^2}(A \log r + B)_{\theta\theta} \\ &= \frac{-A}{r^2} + \frac{1}{r}\frac{A}{r} \\ &= 0,\end{aligned}$$

and so u is harmonic. Solving $A \log r_1 + B = 1, A \log r_2 + B = 0$ gives

$$A = \left(\log \frac{r_1}{r_2}\right)^{-1} \quad , \quad B = -\log R \left(\log \frac{r_1}{r_2}\right)^{-1}.$$

The polar form of the Dirichlet integral can be found, by a simple change of variables, to be

$$\iint_{\mathcal{D}} \left[(u_r)^2 + \frac{1}{r^2}(u_\theta)^2 \right] \, r \, dr \, d\theta.$$

Thus,

$$\begin{aligned}
I_{A(r_1,r_2)}\,[u] &= \int_{r_1}^{r_2} \int_{-\pi}^{\pi} \left[(u_r)^2 + \frac{1}{r^2}(u_\theta)^2 \right] \, r d\theta \, dr \\
&= \int_{-\pi}^{\pi} \int_{r_1}^{r_2} \frac{A^2}{r^2} \, r dr \, d\theta \\
&= 2\pi \left(\log \frac{r_1}{r_2} \right)^{-2} (\log r_2 - \log r_1) \\
&= 2\pi \left(\log \frac{r_2}{r_1} \right)^{-1} \qquad (1.24)
\end{aligned}$$

Chapter 2

Symmetrization and Desymmetrization

In this chapter we will become familiar with some of the ideas needed to prove the extremal properties of harmonic measures of the domains in question. The two fundamental ideas are that of symmetrization and desymmetrization. We begin with symmetrization. As the domain of interest is circular in nature the symmetrization we will be interested in will be circular, or Pölya, symmetrization. The other well known symmetrization is Steiner symmetrization which is linear in nature. A discussion of both can be found in [10].

2.1 Circular Symmetrization

There are two types of circular symmetrization that will be considered in this section. The first is circular symmetrization of domains and the second, symmetrization of functions. The later is defined using the former, so we look at domain symmetrization first. These concepts were introduced around the middle of the century and developed by Pólya and Szegö to investigate, among other things, bounds on coefficents of expansions of regular functions. We will need these ideas to allow us to compare functions on domains described in the introduction differing only in the positioning of the slits. As such, the material covered in this section is chosen for its relevence to this aim. Hayman's book, [10], covers the theorems here and more, as well as discussing a linear form of symmetrization due to Steiner.

Let $\mathcal{D}$ be a domain in the complex plane. We define the *circular symmetrization* of the domain $\mathcal{D}$, and denote it $\mathcal{D}^\sharp$, as follows. Firstly, 0 is in $\mathcal{D}^\sharp$ if and only if 0 is in $\mathcal{D}$, and similarly for ∞. If the intersection of $\mathcal{D}$ with C_r is C_r or $\emptyset$ then the intersection of $\mathcal{D}^\sharp$ with C_r is to be C_r or $\emptyset$ respectively. Otherwise $\mathcal{D} \cap C_r$ is a set of open arcs with total length

rl_r and we ask that $\mathcal{D}^\sharp$ meet the circle C_r in the single arc

$$\{re^{i\theta} : |\theta| < \frac{1}{2}l_r\}.$$

It should be clear from this definition that $\mathcal{D}^\sharp$ is symmetric on each circle in the complex plane about the positive real axis. Also, $\mathcal{D}^\sharp$ and $\mathcal{D}$ intersect C_r in sets of the same total arc length. A slightly less trivial observation is the following.

Proposition 2.1.1. *Using the above notation, if $\mathcal{D}$ is a domain, then $\mathcal{D}^\sharp$ is a domain.*

Proof (as in [10]). Let $\mathcal{D}$ be a domain. Suppose the circles C_{r_1} and C_{r_2} intersect $\mathcal{D}^\sharp$ in a non-empty set. Then the whole interval $[r_1, r_2]$ on the real line must be in $\mathcal{D}^\sharp$ as if it weren't there would be a $r \in [r_1, r_2]$ such that the intersection of C_r with $\mathcal{D}$ is empty. This would mean $\mathcal{D}$ could be expressed as the disjoint union of open sets one lying in $\{z : |z| < r\}$ and the other in $\{z : |z| > r\}$. Now any two points $(r_1, \theta_1), (r_2, \theta_2)$ in $\mathcal{D}^\sharp$ can be connected by the curve defined piecewise with endpoints $r_1e^{i\theta_1}$ and r_1 on C_{r_1}, r_1 and r_2 on $\mathbb{R}$, r_2 and $r_2e^{i\theta_2}$ on C_{r_2}. Hence $\mathcal{D}^\sharp$ is connected.

To show $\mathcal{D}^\sharp$ is open consider a point $(r_0, \theta_0) \in \mathcal{D}^\sharp$. We want to show we can place an open ball about (r_0, θ_0).

If $\theta_0 = \pi$ then the entire circle C_{r_0} lies in $\mathcal{D}^\sharp$ and so C_{r_0} lies entirely in $\mathcal{D}$, by the definition of $\mathcal{D}^\sharp$. Since $\mathcal{D}$ is open we can place an open annulus $\mathcal{A}(r_0-\delta, r_1+\delta)$ in $\mathcal{D}$ and hence in $\mathcal{D}^\sharp$. This implies (r_0, π) is an interior point. We use a similar idea to show (r_0, θ_0) is an interior point for $|\theta_0| < \pi$. As we assume (r_0, θ_0) is in $\mathcal{D}^\sharp$ it must lie in $\{(r_0, \theta) : |\theta| < \frac{1}{2}l_{r_0}\} \subset C_{r_0}$ where l_r is as in the definition of $\mathcal{D}^\sharp$. Choose a $l \in (2|\theta_0|, l_{r_0})$. The intersection of $\mathcal{D}$ and C_{r_0} consists of $N < \infty$ open intervals of total length $r_0l_{r_0}$. We choose a collection of closed arcs $\{[\alpha_n, \beta_n] : 1 \le n \le N\}$ (taken as a subset of $\mathcal{D} \cap C_{r_0}$) with total length greater than r_0l. Let δ_n be the minimum distance of the nth arc from the boundary of $\mathcal{D}$ and set $\delta = \min\{\delta_n : 1 \le n \le N\}$. The sets about these closed arcs defined by

$$O_n = (r_0 - \delta, r_0 + \delta) \times [\alpha_n, \beta_n]$$

are contained in $\mathcal{D}$ for each $1 \le n \le N$. For each $r \in (r_0 - \delta, r_0 + \delta)$ the intervals $[\alpha_n, \beta_n]$ lie completely within C_r with total length l. Hence for each of these r,

$$l_r \ge l > 2|\theta_0|$$

and so

$$(r_0, \theta_0) \in (r_0 - \delta, r_0 + \delta) \times (-\frac{1}{2}l, \frac{1}{2}l) \subset \mathcal{D}^\sharp.$$

Therefore (r_0, θ_0) is an interior point of $\mathcal{D}^\sharp$. As this point was arbitrary it follows that $\mathcal{D}^\sharp$ is open and hence a domain. □

We now wish to use domain symmetrization to define a way of constructing symmetric functions.

Consider an arbitrary domain $\mathcal{D} \subseteq \mathbb{C}$ and the characteristic function $\chi_{\mathcal{D}}$ on this domain. If we symmetrize $\mathcal{D}$ to obtain $\mathcal{D}^\sharp$, the characteristic function, $\chi_{\mathcal{D}^\sharp}$, is circularly symmetric about the positive real axis. That is,

$$\chi_{\mathcal{D}^\sharp}(re^{i\theta}) = \chi_{\mathcal{D}^\sharp}(re^{-i\theta})$$

for all $\theta \in [-\pi, \pi]$ and $r \in [0, \infty)$. It is this idea that will motivate our definition of the symmetrization of a function.

Definition 2.1.2. Let $f : \mathbb{C} \to \mathbb{R}$ be a bounded, continuous function. The *distribution sets, $\{\mathcal{D}_t : -\infty < t < \infty\}$, of f* are defined by

$$\mathcal{D}_t = \{z \in \mathbb{C} : f(z) > t\} \qquad t \in (-\infty, \infty). \tag{2.1}$$

These sets will enable us to get a handle on the distribution of f over $\mathbb{C}$. By "chopping off" the part of f lying at or below t a similar process to that carried out on the characteristic function can be performed. It is tempting to remove the boundedness and continuity restrictions in the above definition and talk of more general distribution sets. However, the functions encountered in this thesis will generally satisfy these conditions which give us the following useful properties.

Proposition 2.1.3. *Let $\mathcal{D}_t$ denote the distribution sets of a bounded, continuous, real-valued function f. Then,*

1. *$\mathcal{D}_t$ is open for each $t \in (-\infty, \infty)$.*
2. *The sets $\mathcal{D}_t$ form a decreasing family of subsets of $\mathbb{C}$. That is, $\mathcal{D}_s \supseteq \mathcal{D}_t$ whenever $s \leq t$.*
3. *The $\mathcal{D}_t$ are* concentrated *on some interval $[T_1, T_0] \subset (-\infty, \infty)$. That is,*

$$\begin{aligned} \mathcal{D}_t &= \emptyset \qquad \text{for all } t > T_0 \\ \mathcal{D}_t &= \mathbb{C} \qquad \text{for all } t < T_1 \end{aligned}$$

Proof. The first statement is trivial as f is assumed to be continuous, and $\mathcal{D}_t$ is just the pre-image of the open set (t, ∞). If $s \leq t$ then $(s, \infty) \supseteq (t, \infty)$ and so

$$\mathcal{D}_s = f^{-1}((s, \infty)) \supseteq f^{-1}((t, \infty)) = \mathcal{D}_t$$

proving the second assertion. The boundedness assumption for f means there is a $M < \infty$ such that for $z \in \mathbb{C}, |f(z)| \leq M$. Let $T_1 = -M, T_0 = M$. Then if $t < T_1 = -M$ we have that $f(z) > t$ for all $z \in \mathbb{C}$ and so $\mathcal{D}_t = \mathbb{C}$. If $t > T_0 = M$ then there are no $z \in \mathbb{C}$ for which $f(z) > t$ and so $\mathcal{D}_t = \emptyset$. □

This proposition tells us that the $\mathcal{D}_t$ "grow" from the empty set to the entire plane as we let t get smaller. The boundaries of these sets are the level sets of f, ie. $f(z) = t$ for all $z \in \partial\mathcal{D}_t$. If we watch where these points on the boundary get mapped to under symmetrization we can construct a symmetrized version f slice by slice. This is essentially what is captured in the following defintion.

Definition 2.1.4. Let f be as in Definition 2.1.2. Then we denote by $f^\sharp$ the *symmetric decreasing rearrangement of* f, defined by

$$f^\sharp(z) = \sup\{t : z \in \mathcal{D}_t^\sharp\}.$$

If f is a continuous function on a compact subset, E, of $\mathbb{C}$ it is known that f must be uniformly continuous on E (see, for example [2]). That is,

$$|f(z_1) - f(z_2)| \leq P(\delta)|z_1 - z_2|$$

whenever $|z_1 - z_2| < \delta$. The quantity $P(\delta)$ then vanishes as $\delta \to 0$ an is called the *modulus of continuity* of f. From this definition we can see that f is Lipschitz on E if and only if there exists a C such that $P(\delta) \leq C\delta$ for all $\delta > 0$. Hayman, in [10], gives easily modifiable arguments can be used to prove that circular symmetrization decreases the modulus of continuity. This allows us to prove the following.

Proposition 2.1.5. *Suppose f is a real valued function that is admissible on $\mathcal{D}$. Then $f^\sharp$ is admissible on $\mathcal{D}^\sharp$.*

Sketch of proof. To show $f^\sharp$ is admissible we want to show that it is Lipschitz in $\mathcal{D}$ and C^1 on all but a finite number of points or smooth arcs. By the result that symmetrization decreases the modulus of continuity we see that if f is Lipschitz then $\mathcal{D}^\sharp$ must also be Lipschitz. Suppose that f is C^1 in the open neighbourhood of a point $z_0 = r_0 e^{i\theta_0}$ and let w_0 be the value of its derivative at this point. Since f' is continuous it will map open sets about w_0 to open sets containing z_0. Proposition 2.1.1 and the decreasing modulus of continuity is then enough to prove this result. □

The next theorem plays a crucial role in proving Dubinin's theorem. It is a modification of the Pólya-Szegö symmetrization principle as presented in [10].

Theorem 2.1.6. *Let $\mathcal{A}(\varepsilon) = \mathcal{A}(\varepsilon, 1)$ and $f : \mathcal{A}(\varepsilon) \to [0, 1]$ be admissible on $\mathcal{A}(\varepsilon)$. Also, suppose for each $\varepsilon < r < 1, f_\theta(re^{i\theta}) = 0$ only for isolated points on C_r. Then,*

$$I_{\mathcal{A}(\varepsilon)}\left[f^\sharp\right] \leq I_{\mathcal{A}(\varepsilon)}\left[f\right].$$

Proof. Recalling the polar form of the Dirichlet integral shown in Section 1.4 we have

$$I_{\mathcal{A}(\varepsilon)}\left[f\right] = \int_\varepsilon^1 \int_{-\pi}^{\pi} \left(f_r^2 + \frac{1}{r^2} f_\theta^2\right) r\, d\theta\, dr.$$

For each $r \in (\varepsilon, 1)$ let

$$J(r) = \int_{-\pi}^{\pi} \left(f_r^2 + \frac{1}{r^2} f_\theta^2 \right) \, d\theta$$

and a_m be the values of $f(re^{i\theta})$ for which $f_\theta(re^{i\theta}) = 0$. Notice that there can only be finitely many of these points as if there weren't they would cluster somewhere on C_r contradicting the assumption that they are isolated points. Let $b_1, \ldots, b_N$ be the values of $f(re^{i\theta})$ for which $f_\theta(re^{i\theta})$ does not exist. Let t_m be the collection of the a_m's, the b_m's and the points 0 and 1 arranged in increasing order with m. Consider two successive values from this sequence, $t_m < t_{m+1}$. By the continuity of f there are k_m open intervals on C_r where f is increasing between t_m and t_{m+1} and k_m open intervals on which f is decreasing. Also these intervals appear alternately in C_r. Let the family $T_{m,\nu}$ denote these intervals as they appear on C_r, with f decreasing for even ν, decreasing for odd ν, $\nu = 1, \ldots, 2k_m$. As we vary over m it is easy to see that the $T_{m,\nu}$ cover all of C_r except the points t_m. Therefore,

$$J(r) = \sum_m \sum_{\nu=1}^{2k_m} \int_{T_{m,\nu}} \left(f_r^2 + \frac{1}{r^2} f_\theta^2 \right) \, d\theta.$$

We want to now change the variable of integration from θ to $t = f(re^{i\theta})$. As f is strictly increasing or decreasing on $T_{m,\nu}$ the relationship between t and f on these intervals is one-to-one. Hence, we can write

$$\theta = \theta_\nu(t, r) \qquad (1 \le \nu \le 2k_m)$$

and consider θ as a function of t and r in $T_{m,\nu}$. We now calculate the partial derivatives of θ, keeping in mind that $t = f(re^{i\theta})$ on each $T_{m,\nu}$.

$$\frac{\partial f}{\partial \theta} = \left(\frac{\partial \theta}{\partial t} \right)^{-1} \quad , \quad \frac{\partial f}{\partial r} = - \left(\frac{\partial \theta}{\partial r} \right) \left(\frac{\partial \theta}{\partial t} \right)^{-1} \quad , \quad d\theta = \frac{\partial \theta}{\partial t} \, dt.$$

Substituting this back into $J(r)$ gives

$$J(r) = \sum_m \int_{t_m}^{t_{m+1}} \sum_{\nu=1}^{2k_m} \left(\frac{(\theta_\nu)_r^2}{|(\theta_\nu)_t|} + \frac{1}{r^2} \frac{1}{|(\theta_\nu)_t|} \right) \, dt. \tag{2.2}$$

For $t_m < t < t_{m+1}$ the values of θ for which $f(re^{i\theta}) > t$ are $\theta_{2\nu-1} < \theta < \theta_{2\nu}, (1 \le \nu \le k_m)$. Hence the length of $\{\theta \in [-\pi, \pi] : f(re^{i\theta}) > t\}$ is $rl_r(t)$, where

$$l_r(t) = \sum_{\nu=1}^{k_m} (\theta_{2\nu}(t) - \theta_{2\nu-1}(t)).$$

It can be seen that $l_r(t)$ is strictly decreasing as t increases in (t_m, t_{m+1}). Let $\phi(t) = \pm\frac{1}{2} l_r(t)$. Then,

$$f^\sharp(re^{i\phi}) = t$$

for $t_m < t < t_{m+1}$ by the definition of $f^\sharp$. Using similar arguments to those for $J(r)$ we see that the inner part of the Dirichlet integral for $f^\sharp$ is given by

$$J^\sharp(r) = \sum_m \int_{t_m}^{t_{m+1}} 2\left(\frac{\phi_r^2}{|\phi_t|} + \frac{1}{r^2}\frac{1}{|\phi_t|}\right) dt \tag{2.3}$$

where

$$\phi(t) = \frac{1}{2}\left[(\theta_2(t) - \theta_1(t)) + \cdots + (\theta_{2k_m}(t) - \theta_{2k_m-1}(t))\right].$$

Looking at the partial derivatives of ϕ reveals

$$\frac{\partial\phi}{\partial t} = \frac{1}{2}\sum_{\nu=1}^{2k_m}(-1)^\nu \frac{\partial\theta_\nu}{\partial t} = -\frac{1}{2}\sum_{\nu=1}^{2k_m}|(\theta_\nu)_t|,$$

and,

$$\frac{\partial\phi}{\partial r} \leq \frac{1}{2}\sum_{\nu=1}^{2k_m}|(\theta_\nu)_r|.$$

We now apply the arithmetic-harmonic mean inequality to obtain

$$\frac{2}{\phi_t} = \frac{4}{\sum_1^{2k_m}|(\theta_\nu)_t|} \leq \frac{4}{(2k_m)^2}\sum_1^{2k_m}\frac{1}{(\theta_\nu)_t}.$$

A simple manipulation of the Cauchy-Schwarz inequality gives us

$$\left(\sum a\right)^2 \leq \left(\sum \frac{a^2}{|b|}\right)\left(\sum |b|\right),$$

and so

$$2\frac{\phi_r^2}{|\phi_t|} \leq \frac{\left(\sum|(\theta_\nu)_r|\right)^2}{\sum|(\theta_\nu)_t|} \leq \sum \frac{|(\theta_\nu)_r|^2}{|(\theta_\nu)_t|}.$$

Substituting these two inequalities back into (2.2) and (2.3) shows that for each $r \in (\varepsilon, 1)$,

$$J^\sharp(r) \leq J(r)$$

proving the theorem. □

In order to prove some results in the next section we will need some results about symmetrization of functions of a real variable, namely continuous, and therefore bounded, functions $f : [-\pi, \pi] \to \mathbb{R}$. As in the preliminaries, we identify intervals of length 2π with circles of arbitrary non-zero radius. This means the functions under scrutiny will have to satisfy $f(-\pi) = f(\pi)$ to be continuous on the interval $[-\pi, \pi]$. A symmetric decreasing rearrangement of these real variable functions is defined analogously to the complex variable case. The sets $\mathcal{D}_t$ are now open subsets of $[-\pi, \pi]$ with Lebesgue measure l_t. The sets $\mathcal{D}_t^\sharp$ are then the open intervals

$$\mathcal{D}_t^\sharp = \{x \in [-\pi, \pi] : |x| < \frac{1}{2} l_t\} \qquad (-\infty < t < \infty)$$

and we define $f^\sharp$ by

$$f^\sharp(x) = \sup\{t : x \in \mathcal{D}_t^\sharp\} \qquad x \in [-\pi, \pi].$$

The statements about the distribution sets in Proposition 2.1.3 are equally valid for these $\mathcal{D}_t$. It is fairly obvious that if we are given a function and we symmetrically rearrange it on each circle C_r in $\mathbb{C}$ then the function we obtain will be precisely the symmetric decreasing rearrangement of a complex variable. This reason for this is that circular symmetrization of a complex variable was defined using precsiely this method.

Definition 2.1.7. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a real valued, integrable function on $[-\pi, \pi]$. Denote by λ_f the *distribution function of* f given by

$$\lambda_f(t) = |\{x : f(x) > t\}| = |\mathcal{D}_t| \tag{2.4}$$

The two main results we will need about distributions is that when we symmetrize a function f its distribution is invariant. This comes straight from the definition of a symmetric rearrangement in terms of distribution sets. Also, given any $\inf f < c < \sup f$ we can find a $t \in (-\infty, \infty)$ such that

$$\lambda_f(t^+) \leq c \leq \lambda_f(t^-).$$

By t^+ and t^- we mean the limit approaching t from above and below respectively.

We now extend the idea of a symmetric decreasing rearrangement by rearranging a function about n evenly spaced points in $[-\pi, \pi]$.

Definition 2.1.8. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a real-valued, continuous function and $n \geq 1$ an integer. The function $g : [-\pi, \pi] \to \mathbb{R}$ defined by

$$g(x) = f^\sharp(nx) \qquad (x \in [-\pi, \pi])$$

is called the *n-fold symmetric decreasing rearrangement of* f or more concisely the n-fold rearrangement of f.

A 1-fold symmetric decreasing rearrangement is clearly just a symmetric decreasing rearrangement. As discussed earlier we can extend these rearrangements of a real variable to rearrangements of functions of a complex variable. We can therefore define a n-fold symmetric decreasing rearrangement of a function $f : \mathbb{C} \to \mathbb{R}$ by rearranging $f(re^{i\theta})$ as a function of θ on each circle C_r in $\mathbb{C}$. We need this idea to be able to take a function defined on a uneven radially slit disk Ω and move its boundary values so that they can be compared with those of $\tilde{\Omega}$. By a simple modification of the proof of Theorem 2.1.6 we can show the following result.

Proposition 2.1.9. *Suppose $f : \mathcal{D} \to [0,1]$ is an admissible function. Furthermore, suppose that f takes on the value 0 at exactly n distinct points on each circle C_r. We also require f take on the value 1 at n distinct points on each circle. Then,*

$$I_{\mathcal{D}}[g] \leq I_{\mathcal{D}}[f]$$

where g is the n-fold symmetric decreasing rearrangement of f.

The n-fold arrangements presented in this section allow us to redistribute a function about n evenly spaced points. In the next section we shall see a method, developed by Dubinin, that allows us to redistribute a function about n points that are not evenly spaced.

2.2 The $*$-operator

As we have seen in the previous section, circular symmetrization gives us a way of comparing functions on domains such as our Ω and $\tilde{\Omega}$. Although useful in the proof of Theorem 1.1.3 (as we shall see), it only allows us to compare the functions in a pointwise manner. In Theorem 1.1.4 we are looking at the harmonic measures on Ω and $\tilde{\Omega}$ as a whole — integrating them over the entire domain as the argument of a quite arbitrary function. We call the result of this integration an *integral mean.*

Definition 2.2.1. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a convex, non-decreasing function. Then for $f \in L^1[a,b]$ we define the *integral mean of Φ with respect to f* as

$$\int_a^b \Phi(f(x))\, dx.$$

We can see straight away that this generalizes the $u(0) \leq v(0)$ inequality of Theorem 1.1.3 by choosing Φ to be the identity on $[0,1]$ and using the mean value property. The cost of this generalization is that even for the case when the domains Ω and $\tilde{\Omega}$ are simply connected, as in Dubinin's theorem, we are unable to show that the integral mean inequalities hold when we have four or more slits. Even though the proof is elusive, it is conjected by Baernstein in [5] that this is in fact the case. Lending credence to this conjecture are some computational results discussed in Chapter 4 based on ideas of Quine's in [16].

As integral means turn out to be reasonably hard to manipulate, Baernstein introduces what he calls $*$-functions. The definitions and proofs given in this section are as in [3]. The paper this section is based on developed the star function notation to prove some extremal properties of the Koebe function and integral means. This was nine years before Dubinin proved his theorem and twelve before the proof of Theorem 1.1.4 appeared. Baernstein saw that his statement was a fairly natural generalization of Dubinin's and applied the theory presented below.

Definition 2.2.2. For each $f \in L^1[a,b]$ we define the $*$-*function* of f by

$$f^*(l) = \sup_{|E|=2l} \int_E f(x) \tag{2.5}$$

for each $l \in \left[0, \frac{b-a}{2}\right]$. The integral is with respect to Lebesgue measure on $[a,b]$ and the supremum is taken over all Borel subsets E, of $[a,b]$, with measure $2l$.

Our main aim in this section is to show that for u,v and Φ as in Theorem 1.1.4, the inequalities

$$\int_{-\pi}^{\pi} \Phi(u(re^{i\theta}))\,d\theta \le \int_{-\pi}^{\pi} \Phi(v(re^{i\theta}))\,d\theta \qquad (0 < r < 1)$$

$$u^*(re^{i\theta}) \le v^*(re^{i\theta}) \qquad (re^{i\theta} \in \Delta^+)$$

are equivalent regardless of the choice of Φ. The functions u^* and v^* are defined on the upper half-disk $\Delta^+ = \Delta \cap \{x + iy \in \mathbb{C} : y > 0\}$ by considering $u(re^{i\theta})$ and $v(re^{i\theta})$ as real valued functions of $\theta \in [0, \pi]$ for each fixed $r \in (0,1)$. Also covered here are any other properties of $*$-functions that will be needed in the proof of Baernstein's theorem.

Proposition 2.2.3. *For each $l \in \left[0, \frac{b-a}{2}\right]$ there exists a measurable set $E \subseteq \left[0, \frac{b-a}{2}\right]$ with $|E| = 2l$ such that the supremum in 2.5 is obtained.*

Proof. Let f be as in Definition 2.2.2. As we can choose $E = \emptyset$ and $E = \left[0, \frac{b-a}{2}\right]$ for $l = 0, \frac{b-a}{2}$ respectively we will only consider $l \in \left(0, \frac{b-a}{2}\right)$. Let $\lambda(t) = |\{x : f(x) > t\}|$ be the distribution of f, as discussed in the previous section. There exists, a $t \in (-\infty, \infty)$ such that

$$\lambda(t^+) = \lambda(t) \le 2l \le \lambda(t^-).$$

Let $A = \{x : f(x) > t\}$ and $B = \{x : f(x) \ge t\}$. We can then find a measurable E such that $A \subseteq E \subseteq B$ and $|E| = 2l$ as $|A| = \lambda(t)$ and $|B| = \lambda(t^-)$. Suppose F is any other measurable set with $|F| = 2l$. Then,

$$\begin{aligned}\int_F f\,dx &= \int_F (f(x) - t)\,dx + \int_F t\,dx \\ &= \int_F (f(x) - t)\,dx + 2lt \\ &\le \int_a^b [f(x) - t]^+\,dx + 2lt \end{aligned} \tag{2.6}$$

where

$$[g(x)]^+ \overset{\text{def}}{=} \begin{cases} 0 & \text{if } g(x) < 0, \\ g(x) & \text{otherwise.} \end{cases} \tag{2.7}$$

The inequality arises in (2.6) as $(f(x)-t)$ could be negative for $x \in F$, whereas $[f(x)-t]^+$ is non-negative on $[a,b]$ by definition. Notice that as $E \subseteq B$, $f(x)-t \geq 0$ on E and hence equal to $[f(x)-t]^+$. Thus,

$$\int_a^b [f(x)-t]^+ \, dx + 2lt = \int_E (f(x)-t) \, dx + 2lt = \int_E f \, dx \tag{2.8}$$

as $|E| = 2l$. Combining (2.6) and (2.8) shows that

$$\int_F f \, dx \leq \int_E f \, dx$$

and as F was an arbitrary set with $|F| = 2l$, E is in fact a set for which the supremum is attained. $\square$

Intuitively, the $*$-function is obtained by arranging the original function from highest to lowest and "summing" over the first $2l$ "points". We have already seen how to do such rearrangements of functions in Section 2.1. This next proposition shows exactly what is meant by this explanation.

From here on we will restrict the interval that our functions are defined on to $[-\pi,\pi]$ as we can identify this interval with points on a circle. The proofs given can be easily modified for arbitrary intervals by simply translating and/or dilating $[-\pi,\pi]$.

Proposition 2.2.4. *Suppose $f \in L^1[-\pi,\pi]$ and that $f^\sharp$ is the symmetric, non-increasing rearrangement of f. Then the relation between f^* and $f^\sharp$ is given by*

$$f^*(l) = \int_{-l}^{l} f^\sharp(x) \, dx \qquad (0 \leq l \leq \pi) \tag{2.9}$$

Proof. It is clear that the above equation holds for $l = 0$ and $l = \pi$ so we will focus our attention on the case when $l \in (0,\pi)$. We know that there exist a subset E of $[-\pi,\pi]$ for which

$$f^* = \int_E f(x) \, dx$$

by Proposition 2.2.3. Let t be the one used in the proof of Proposition 2.2.3. Then,

$$f^* = \int_E f(x) \, dx = \int_{-\pi}^{\pi} [f(x)-t]^+ \, dx + 2lt. \tag{2.10}$$

As f and $f^\sharp$ have the same distribution, $[f(x)-t]^+$ and $[f^\sharp(x)-t]^+$ will also. This means the last integral in (2.10) can be replaced by $\int_{-\pi}^{\pi}[f^\sharp(x)-t]^+ \, dx$ giving

$$f^* = \int_{-\pi}^{\pi} [f^\sharp(x)-t]^+ \, dx + 2lt.$$

By virtue of its construction, the distribution sets, $\{x : f^\sharp(x) > s\}$, of $f^\sharp$ are symmetric about the origin. Since $|\{x : f^\sharp(x) > t\}| = |\{x : f(x) > t\}| = |E| = 2l$ we have $\{x : f^\sharp(x) > t\} = (-l, l)$. So for $|x| < l, f^\sharp(x) > t$ and for $|x| \geq l$ we see $f^\sharp(x) \leq t$. Coupling these facts with the definition of $[f^\sharp(x) - t]^+$ gives

$$f^* = \int_{-l}^{l} (f^\sharp(x) - t)\, dx + 2lt = \int_{-l}^{l} f^\sharp(x)\, dx$$

proving the proposition. □

This gives us enough machinery to prove the main theorem of this section. In the statement below we are really only concerned with the equivalence of the first and third inequalities. The second inequality is used as a stepping stone but is an interesting fact in its own right.

Theorem 2.2.5. *Let $f, g \in L^1[-\pi, \pi]$ be real valued functions. Then the following statements are equivalent.*
(a) For every convex, non-decreasing function Φ on $(-\infty, \infty)$,

$$\int_{-\pi}^{\pi} \Phi(f(x))\, dx \leq \int_{-\pi}^{\pi} \Phi(g(x))\, dx.$$

(b) For all $t \in (-\infty, \infty)$,

$$\int_{-\pi}^{\pi} [f(x) - t]^+\, dx \leq \int_{-\pi}^{\pi} [g(x) - t]^+\, dx.$$

(c) For each $x \in [0, -\pi]$,

$$f^*(x) \leq g^*(x)$$

Proof. To prove (a) implies (b) we observe that for every $t \in (-\infty, \infty)$ the function $\Phi(x) = [x - t]^+$ is a convex, non-decreasing function on $(-\infty, \infty)$ and the inequality of (b) follows directly from (a). Now assume (b) and let $t = g^\sharp(\theta)$ for some $0 \leq \theta \leq \pi$. Suppose $E \subset [-\pi, \pi]$ such that $|E| = 2\theta$. Then

$$\begin{aligned}\int_E f = \int_E [f(x) - t]\, dx + 2\theta t &\leq \int_{-\pi}^{\pi} [f(x) - t]^+\, dx + 2\theta t \\ &\leq \int_{-\pi}^{\pi} [g(x) - t]^+\, dx + 2\theta t\end{aligned}$$

by the assumption of (b). Since g and $g^\sharp$ have the same distribution,

$$\int_{-\pi}^{\pi} [g(x) - t]^+\, dx = \int_{-\pi}^{\pi} [g^\sharp(x) - t]^+\, dx = \int_{-\theta}^{\theta} g^\sharp(x)\, dx - 2\theta t$$

Using this and Proposition 2.2.4 we get

$$\int_E f \leq= \int_{-\theta}^{\theta} g^\sharp(x)\,dx = g^*(\theta)$$

for any set E with measure 2θ. As $f^*(\theta)$ is the supremum over all such E, $f^*(\theta) \leq g^*(\theta)$ for $0 \leq \theta \leq \pi$, proving (c). To prove (c) implies (b) we will need the notion of *essential infimum* (ess. inf) and *essential supremum* (ess. sup). The adjective essential basically means "off a set of measure zero", so if $m = \text{ess.}\sup f$ over $[a, b]$ then $|\{x \in [a, b] : f(x) > m\}| = 0$, with $|\cdot|$ denoting Lebesgue measure. Similarly for ess. inf. Now assume that $t < \text{ess.}\sup f$ and choose θ so that

$$f^\sharp(\theta^-) \geq t \geq f^\sharp(\theta^+).$$

If $t \geq \text{ess.}\sup f$ then choose $\theta = 0$ and for $t \leq \text{ess.}\inf f$ let $t = \pi$. This way we are ignoring "outliers" of f which do not get noticed when comparing integral inequalities. Now,

$$\begin{aligned}\int_{-\pi}^{\pi} [f(x) - t]^+\,dx &= \int_{-\pi}^{\pi} [f^\sharp(x) - t]^+\,dx \\ &= \int_{-\theta}^{\theta} [f^\sharp(x) - t]\,dx \\ &= \int_{-\theta}^{\theta} f^\sharp(x)\,dx - 2\theta t \\ &= f^*(\theta) - 2\theta t \\ &\leq g^*(\theta) - 2\theta t\end{aligned}$$

since we have assumed $f^* \leq g^*$ for (c). By Proposition 2.2.4

$$\begin{aligned}g^*(\theta) - 2\theta t &= \int_{-\theta}^{\theta} [g^\sharp(x) - t]\,dx \\ &= \int_{-\pi}^{\pi} [g^\sharp(x) - t]^+\,dx \\ &= \int_{-\pi}^{\pi} [g(x) - t]^+\,dx\end{aligned}$$

proving (b). Finally, we assume (b) and show that (a) follows. Suppose Φ is a non-decreasing, convex function and furthermore $\Phi(x) = 0$ for $x \in (-\infty, -M)$, where $M < \infty$. The convexity of Φ implies that $\Phi'(x)$ is non-decreasing on $(-\infty, \infty)$ and so there is a positive measure, μ on $(-\infty, \infty)$ defined by

$$\mu(-\infty, s) = \Phi'(s^-).$$

Now,

$$\Phi(s) = \int_{-M}^{s} \Phi'(t)\,dt = \int_{-\infty}^{s} \Phi'(t)\,dt$$

since we have assumed $\Phi(x) = 0$ for $x < -M$. By an integration by parts we get

$$\int_{-\infty}^{s} \Phi'(t)\,dt = -\int_{-\infty}^{s} \Phi'(t)\,d(s-t) = \int_{-\infty}^{s} (s-t)\,d\mu(t)$$

with the last integral due to the definition of μ as the measure induced by Φ'. As $[s-t]^+ = 0$ for $t \geq s$ we have

$$\Phi(s) = \int_{-\infty}^{\infty} [s-t]^+\,d\mu(t)$$

for all $-\infty < s < \infty$. Using this representation we can write

$$\begin{aligned}\int_{-\pi}^{\pi} \Phi(f(x))\,dx &= \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} [f(x)-t]^+\,d\mu(t)\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} [f(x)-t]^+\,dx\,d\mu(t)\end{aligned}$$

by Fubini as everything is in $L^1[-\pi, \pi]$. By our assumption of (b), and Fubini again, we have

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\pi}^{\pi} [f(x)-t]^+\,dx\,d\mu(t) &\leq \int_{-\infty}^{\infty}\int_{-\pi}^{\pi} [g(x)-t]^+\,dx\,d\mu(t) \\ &= \int_{-\pi}^{\pi}\int_{-\infty}^{\infty} [g(x)-t]^+\,d\mu(t)\,dx \\ &= \int_{-\pi}^{\pi} \Phi(g(x))\,dx.\end{aligned}$$

This proves the implication of (a) from (b) for the case that $\Phi(x) = 0$ for $x < -M$. For an arbitrary Φ denote by Φ_n the function

$$\Phi_n(x) = \begin{cases} \Phi(x) & \text{if } x \geq -n, \\ \Phi(-n) & \text{otherwise.} \end{cases}$$

which is convex, non-decreasing for all $n \in \mathbb{N}$, and $\Phi_n \to \Phi$ pointwise on $(-\infty, \infty)$ as $n \to \infty$. Furthermore, the above argument holds for each $n \in \mathbb{N}$ and the function $\Psi(x) = \Phi_n(x) - \Phi(-n)$. Hence,

$$\begin{aligned}\int_{-\pi}^{\pi} \Phi_n(f(x))\,dx &= \int_{-\pi}^{\pi} \Psi(f(x))\,dx + 2\pi\Phi(-n) \\ &\leq \int_{-\pi}^{\pi} \Psi(g(x))\,dx + 2\pi\Phi(-n) \\ &= \int_{-\pi}^{\pi} \Phi_n(g(x))\,dx\end{aligned}$$

for each n. Taking limits of both sides with respect to n proves (a) for arbitrary Φ. □

2.3 Desymmetrization

Presented here will be the notion of desymmetrization of a function. It is this notion, coupled with that of symmetrization, that will be the key idea for comparing functions defined on the symmetric domain $\tilde{\Omega}$ to those defined on Ω. The basic idea here will be to redistribute a given function, in a similar way to n-fold symmetric arrangements, about the points $\alpha_1, \ldots, \alpha_n$. As the name suggests we are taking something symmetric and manipulating it so that it is no longer symmetric in some sense. This restricts us to the following classes of functions.

Definition 2.3.1. For a fixed positive integer n, let

$$\mathcal{S}_n = \{f : \partial\Delta \to \mathbb{R} \mid f(e^{i(\tilde{\alpha}_j+\theta)}) = f(e^{i(\tilde{\alpha}_k+\theta)}), 1 \leq j, k \leq n, \theta \in [\frac{-\pi}{n}, \frac{\pi}{n}]\}$$

where $\tilde{\alpha}_j = \frac{2\pi(j-1)}{n}$ for $j = 1, \ldots, n$.

That is, $\mathcal{S}_n$ is the set of all real-valued functions on the boundary of the unit disc that are symmetric about $e^{i\tilde{\alpha}_j}$ and $e^{i\tilde{m}_j}$, where $\tilde{m}_j = \frac{1}{2}(\tilde{\alpha}_j + \tilde{\alpha}_{j+1}), j = 1, \ldots, n$. In the remainder of this section we will describe a process that, given $\alpha_j \in [0, 2\pi), j = 1, \ldots, n$ and $f \in \mathcal{S}_n$, forms a function $f^\dagger : \partial\Delta \to \mathbb{R}$ which satisfies the conditions in the following lemma.

Lemma 2.3.2. *The desymmetrization $f^\dagger$ of a function $f \in \mathcal{S}_n$ satisfies:*

1. *f and $f^\dagger$ have the same distribution function.*
2. *f and $f^\dagger$ have the same valence. That is, $f(z) = t$ has the same number of solutions as $f^\dagger(z) = t$ for all $t \in \mathbb{R}$.*
3. *If f is continuous (Lipschitz) then so $f^\dagger$ is continuous (Lipschitz).*
4. *$f(e^{i\tilde{\alpha}}) = f(e^{ia_j}), j = 1, \ldots, n$.*

The function $f^\dagger$ is constructed by using a bijection from $\partial\Delta$ to $\partial\Delta$ which rearranges intervals.

We define a family of subsets of $\partial\Delta$ denoted $\{E(\theta)\}_{0\leq\theta\leq\frac{\pi}{n}}$. The sets in this family will have the following properties:

1. $E(0) = \{\alpha_1, \ldots, \alpha_n\}$.
2. $E(\frac{\pi}{n}) = \partial\Delta$.
3. $E(\theta_1) \subseteq E(\theta_2)$ whenever $0 \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{n}$.
4. $E(\theta)$ and $E^c(\theta) = \partial\Delta \setminus E(\theta)$ have n connected components for each $\theta \in [0, \frac{\pi}{n})$.

We will denote the n connected components of $E(\theta)$ and $E^c(\theta)$ by $E_j(\theta)$ and $E^c_j(\theta)$ respectively for $j = 1, \dots, n$. The lengths of these components on $\partial\Delta$ will be $l_j(\theta) = |E_j(\theta)|$, $l^c_j(\theta) = |E^c_j(\theta)|$. Let $l(\theta) = \min_j\{l_j(\theta)\}$ and $l^c(\theta) = \min_j\{l^c_j(\theta)\}$ be the length of the shortest component at each $\theta \in [0, \frac{\pi}{n})$.

The construction of these sets is iterative. Here's the algorithm in pseudo-code:

desym$(\alpha_1, \dots, \alpha_n)$

1) $k \leftarrow 0$

2) $\theta_0 \leftarrow 0$

3) $E(\theta_0) \leftarrow \{\alpha_1, \dots, \alpha_n\}$

4) $b_{j,0} \leftarrow \alpha_j$ for $j = 1, \dots, n$

5) $\delta_{j,0} \leftarrow 0$ for $j = 1, \dots, n$

6) **while** $\theta_k \neq \frac{\pi}{n}$ **do**

7) $\quad k \leftarrow k + 1$

8) $\quad \theta_k \leftarrow \theta_{k-1} + l^c(\theta_{k-1})$

9) $\quad E(\theta) \leftarrow \bigcup_{j=1}^{n} I(b_{j,k}, \delta_{j,k} + (\theta - \theta_{k-1}))$ for $\theta \in (\theta_{k-1}, \theta_k)$

10) $\quad r_k \leftarrow \# \{j : |K^c_j(\theta_{k-1})| = l^c(\theta_{k-1})\}$

11) $\quad E(\theta_{k^-}) \leftarrow \overline{\bigcup_{\theta<\theta_k} E(\theta)}$

12) $\quad \{P_{1,k}, \dots, P_{r_k,k}\} \leftarrow$ **choose**(k, r_k)

13) $\quad E(\theta_k) \leftarrow E(\theta_{k^-}) \cup \{P_{1,k}, \dots, P_{r_k,k}\}$

14) $\quad$ **for** $j = 1$ **to** n **do**

15) $\qquad b_{j,k} \leftarrow$**centre of** $E_j(\theta_k)$

16) $\qquad \delta_{j,k} \leftarrow \frac{l_j(\theta_k)}{2}$

17) $\quad$ **endfor**

18) **endwhile**

A quick discussion of the above code is in order. Lines 1–5 initialize some variables. The $b_{j,k}$ and $\delta_{j,k}$ keep track of the centre and endpoints of the n components of $E(\theta_k)$ when a new θ_k is defined in the main loop (lines 6–18). This main loop terminates when we have completely defined the $E(\theta)$. Inside this loop, the next θ_k is set to the value of the old θ_k plus the half the value of the smallest distance between any of the two intervals in $E(\theta_{k-1})$. On line 9, the $E(\theta)$ are defined for all the θ between the last one and the current one by extending the length of each of the intervals by $\theta - \theta_{k-1}$. On line 10, r_k is the number of intervals inbewteen the ones in $E(\theta)$ that get "squashed" to the empty set as $\theta \to \theta_k$. To satisfy the condition that both $E(\theta)$ and $E^c(\theta)$ consist of n components for all θ, we choose r_k points $P_{1,k}, \dots, P_{r_k,k}$ inbetween the intervals of $E(\theta_{k^-})$ on lines 12 and 13. The **for** loop consisting of lines 14–17 sets the new centres and endpoints of the intervals in $E(\theta_k)$.

Intuitively, what is happening is we are "growing" the sets $E(\theta)$ from the α_j by moving the endpoints of the intervals with velocity 1 until a collision occurs at time θ_1. At this point new intervals are seeded so as to keep the number of intervals at n. These new intervals are then grown at velocity 1 until the next collision time after which new intervals are again seeded and the process continues. With this picture in mind it is not to hard to see that for any θ,

$$|E(\theta)| = 2n\theta. \tag{2.11}$$

When the α_j are evenly spaced the process will terminate after one loop as $\theta_1 = l^c(\theta_0) = \frac{\pi}{n}$. On the other hand the freedom of choice of the $P_{j,k}$ on line 12 makes it unclear that the above algorithm will terminate for arbitrary α_j. In fact, if the α_j are unevenly spaced it is possible to choose the $P_{j,k}$ at each step in such a way that $E(\theta)$ will be undefined for all $\theta \geq l(\theta_0) + \varepsilon$ for any $\varepsilon > 0$. However, for each $E(\theta)$ that is defined, the algorithm ensures that properties 1,3 and 4 on page 38 hold. Property 1 is true by the initialisation on line 3. Property 3 is true as the intervals $I(b_{j,k}, \delta_{j,k} + (\theta - \theta_{k-1}))$ are strictly increasing between the θ_k and at the θ_k all that happens is a closure is taken on line 11 and new points are added. The setting of θ_k to be the minimum distance between any two intervals at the previous step ensures that for all the θ between these steps the $E_j(\theta)$ are disjoint, maintaining property 4. At the θ_k enough points are replaced so as to uphold this property.

To satisfy all the conditions for the $E(\theta)$ on page 38 we only need to ensure that the process terminates after a finite number of steps. This would mean that for some k, $\theta_k = \frac{\pi}{n}$ and so $|E(\theta_k)| = 2n\frac{\pi}{n} = 2\pi$ and hence $E(\frac{\pi}{n}) = \partial\Delta$. As the arguments above for the other properties hold for any θ for which $E(\theta)$ is defined we see that they hold for all $\theta \in [0, \frac{\pi}{n}]$ provided the process terminates.

Lemma 2.3.3. *For the algorithm just discussed there exists, for any initial α_j for $j = 1, \ldots, n$, a choice of the points $\{P_{1,k}, \ldots, P_{r_k,k}\}$ at each step k so that the process terminates in less than $n+1$ steps.*

Proof. The proof is constructive. Here is an algorithm to choose the points $P_{j,k}$ in line 12 of **desym** above.

choose(k, r_k)
1) $L_k \leftarrow |E^c(\theta_{k-})|$
2) $J \leftarrow$Open interval in $E^c(\theta_{k-})$ with $|J| > \frac{1}{n}L_k$
3) $e_k \leftarrow$ An endpoint of J.
4) $P_{1,k} \leftarrow$ The point in J a distance $\frac{1}{n}L_k$ from e_k
5) $J_k(k) \leftarrow$ The interval in $E^c(\theta_{k-})$ with endpoints $P_{1,k}$ and e_k
6) **for** j **from** 1 **to** $k-1$ **do**
7) $\quad$ $J_k(j) \leftarrow$ **update**$(J_{k-1}(j))$
8) **endfor**
9) $\{P_{2,k}, \ldots, P_{r_k,k}\} \leftarrow$ Chosen distinctly from $E^c(\theta_{k-}) \setminus (\bigcup_{j=1}^{k} J_k(j) \cup \{P_{1,k}\})$
10) **return**$(\{P_{1,k}, \ldots, P_{r_k,k}\})$
end

The set $E^c(\theta_{k^-})$ in the above code is $\partial\Delta \setminus E(\theta_{k^-})$. **update** on line 7 moves the endpoints of the intervals it is given toward each other by a distance $\theta_k - \theta_{k-1}$. To make this procedure deterministic it is fairly straightforward to construct a protocol when choosing the points $e_k, P_{1,k}$ on lines 3 and 4, and the rest of the $P_{j,k}$ on line 9. We now show the existence of an interval J as dictated by the conditions on line 2. Assume the contrary, that is, for every interval $E^c_j(\theta_{k^-})$ in $E^c(\theta_{k^-})$, $|E^c_j(\theta_{k^-})| \leq \frac{1}{n}L_k$. Notice that by the way θ_k and r_k are defined in **desym** we have $r_k \geq 1$ and so the number of intervals in $E^c_j(\theta_{k^-})$, $n - r_k \leq n - 1$. Then, as all these intervals are disjoint,

$$\begin{aligned} L_k &= |E^c(\theta_{k^-})| \\ &= \sum_{j=1}^{n-r_k} |E^c_j(\theta_{k^-})| \\ &\leq \sum_{j=1}^{n-r_k} \left(\frac{1}{n}L_k\right) \\ &= (n - r_k)\frac{1}{n}L_k \\ &\leq \frac{n-1}{n}L_k \\ &< L_k \end{aligned}$$

a contradiction. Hence the J in line 2 of **choose** exists. This means when all the $P_{j,k}$ are chosen there will be at least one interval $J_k(k)$ with length $\frac{1}{n}L_k$. This is ensured by line 9 of Choose. We now show that there are in fact k such intervals.

Let $\varepsilon_k = l^c(\theta_k)$, the length of the shortest component in $E^c(\theta_k)$. By the definition of θ_k in Desym as the old θ_k plus half the length of the shortest interval we get:

$$\theta_{k+1} = \theta_k + \frac{1}{2}\varepsilon_k. \tag{2.12}$$

Notice that each interval in $E(\theta_{k-1})$ "grows" by ε_k before reaching $E(\theta_k)$. Also, $absvE^c(\theta_k) = 2\pi - |E(\theta_k)|$ for each k. This, along with the fact that $|E^c(\theta_k)| = |E^c(\theta_{k^-})| = L_k$ gives us the following relation:

$$L_{k+1} = L_k - n\varepsilon_k. \tag{2.13}$$

We know that $|J_k(k)| = \frac{1}{n}L_k$ and is an interval in $E^c(\theta_k)$, hence $\varepsilon_k \leq \frac{1}{n}L_k$. If $\varepsilon_k = \frac{1}{n}L_k$, then 2.13 tells us that $L_{k+1} = 0$ and so $|E(\theta_{k+1})| = 2\pi$ and **desym** would terminate. So proving that **desym** terminates is equivalent to showing that $\varepsilon_k = \frac{1}{n}L_k$ for some k. To see this we look at the sets $J_k(j)$. For induction's sake, let's assume that $|J_{k-1}(j)| = \frac{1}{n}L_{k-1}$ for each $1 \leq j \leq k-1$. When the $J_k(j)$ get defined in line 7 of **choose** the endpoints of

$J_{k-1}(j)$ are moved by $\theta_k - \theta_{k-1}$, which, by (2.12), is equivalent to $\frac{1}{2}\varepsilon_k$. This means the $J_{k-1}(j)$ shrink by $2\frac{1}{2}\varepsilon_k = \varepsilon_k$. So, for each $j = 1, \ldots, k-1$,

$$\begin{aligned} |J_k(j)| &= |J_{k-1}(j)| - \varepsilon_k \\ &= \frac{1}{n} L_{k-1} - \varepsilon_k \\ &= \frac{1}{n} L_k \end{aligned}$$

by the inductive assumption and (2.13). Each $J_k(k)$ is defined to have length $\frac{1}{n}L_k$, and the result is true for $k = 1$ by definition. Hence,

$$|J_k(j)| = \frac{1}{n} L_k \quad \text{for } j = 1, \ldots, k. \tag{2.14}$$

At the kth stage of the process there are k of these J_k each with the above given length. Suppose that the **desym** process has not terminated by step $n-1$, that is, $\theta_{n-1} < \frac{\pi}{n}$. Then by (2.14) there will be $n-1$ intervals $J_{n-1}(j), 1 \leq j \leq n-1$ in $E^c(\theta_{n-1})$ each with length $\frac{1}{n}L_{n-1}$. The nth component in $E^c(\theta_{n-1})$ then has length

$$L_{n-1} - (n-1)\frac{1}{n} L_{n-1} = \frac{1}{n} L_{n-1}$$

also. Hence the length of the shortest interval in $E^c(\theta_{n-1})$, $\varepsilon_{n-1} = \frac{1}{n}L_{n-1}$. Equation (2.13) tells us that $L_n = 0$ and so $|E(\theta_n)| = 2\pi$ and by (2.11) we see that $\theta_n = \frac{2\pi}{2n} = \frac{\pi}{n}$ and the process must terminate.

□

Using the family $E(\theta)$ obtained from the above **desym** algorithm we can construct a map $D : \partial\Delta \to \partial\Delta$. What we want to is match up the endpoints of the expanding intervals in the $E(\theta)$ with the points $\tilde{\alpha}_j \pm \theta$ of the symmetrically expanding intervals for each $\theta \in [0, \frac{\pi}{n}]$. Let θ_k be the collision times of the intervals in the $E(\theta)$ as described above, and assume the Desym process terminates in m steps. That is, $\theta_m = \frac{\pi}{n}$. For each $\theta_{k-1} < \theta < \theta_k, k = 1, \ldots, m$ define for $j = 1, \ldots, n$

$$\begin{aligned} e_j^+(\theta) &= \partial^+ E_j(\theta) \\ e_j^-(\theta) &= \partial^- E_j(\theta) \\ \tilde{e}_j^+(\theta) &= \tilde{\alpha}_j + \theta \\ \tilde{e}_j^-(\theta) &= \tilde{\alpha}_j - \theta \end{aligned}$$

where ∂^+ and ∂^- denote boundaries of an interval in $\partial\Delta$ whose argument is increasing or decreasing respectively. Let $\delta > 0$ be small (less than $\theta_k - \theta_{k-1}$), and define for $k = 1, \ldots, m$

$$\mathcal{C}_k = \{j : \partial^+ E_j(\theta_{k-1} + \delta) \text{ is in a collision at time } \theta_k\}.$$

We set $c_j(\theta_k) \in \partial\Delta, j \in \mathcal{C}_k$ to be the points of collision at time θ_k, precisely

$$c_j(\theta_k) = \lim_{\varepsilon \to 0} \partial^+ E_j(\theta_k - \epsilon)$$

for $j \in \mathcal{C}_k$. At the times θ_k a relabelling of the components of $E(\theta)$ occurs. Let $\sigma_k \in S_n$ denote this relabelling at time θ_k. That is, $E_j(\theta_{k-1}) \mapsto E_{\sigma_k(j)}(\theta_k)$ for $k = 1, \ldots, m-1$ and set σ_0 to the identity for convenience. We keep track of what intervals get mapped to over the course of the construction by putting

$$\tau_k = \sigma_k \circ \cdots \circ \sigma_0, \qquad k = 0, \ldots, m-1.$$

So $\tau_k(j)$ tells us what the interval $E_j(\theta_0) = \alpha_j$ is labelled at the kth stage of **desym** (it also may have collided and restarted as a new interval). We are now ready to define the $e_j^\pm$ at the times θ_k. For $j = 1, \ldots, n$,

$$e_j^-(\theta_k) = \partial^- E_j(\theta_k) \qquad k = 0, \ldots, m-1$$
$$e_j^+(\theta_k) = \begin{cases} \partial^+ E_j(\theta_k) & , \quad j \notin \mathcal{C}_k \\ c_j(\theta_k) & , \quad j \in \mathcal{C}_k \end{cases}$$

Lemma 2.3.4. *Using the notation above the map $D : \partial\Delta \to \partial\Delta$ defined by*

$$D(\tilde{e}_j^+(\theta)) = e_{\tau_k}^+(\theta) \quad \theta \in (\theta_k, \theta_{k+1}]$$
$$D(\tilde{e}_j^-(\theta)) = e_{\tau_k}^-(\theta) \quad \theta \in [\theta_k, \theta_{k+1})$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, m-1$ is a bijection.

Proof. Suppose $z \in \partial\Delta$. As the $E(\theta)$ are closed, increasing with θ, and $E(\frac{\pi}{n}) = \partial\Delta$ there must be a θ^* such that $z \notin E(\theta)$ for $\theta < \theta^*$ and $z \in E(\theta)$ for $\theta \geq \theta^*$. If $\theta_k < \theta^* < \theta_{k+1}$ for some k then $z = e_j^\pm(\theta^*)$ for some j and choice of sign. Then

$$D : \tilde{e}^\pm_{\tau_k^{-1}(j)} \mapsto z.$$

If θ^* is one of the θ_k we have two cases. Either z is one of the c_j's or one of the $P_{j,k}$. If the former is the case

$$D(\tilde{e}^+_{\tau_k^{-1}(j)}(\theta_k)) = e_j^+(\theta_k) = z.$$

In the later situation

$$D(\tilde{e}_{\tau_k}^{-1}(j)^-(\theta_k)) = e_j^-(\theta_k) = z.$$

Hence, D is onto. Notice in the definition of D that $\{\tilde{e}_j^+\} \to \{e_j^+\}$ and $\{\tilde{e}_j^-\} \to \{e_j^-\}$. As the e_j^- and e_j^+ are the endpoints of disjoint intervals or c_j's and $P_{j,k}$'s respectively, $z \in \partial\Delta$ has a uniquely determined endpoint sign. Furthermore, as the $E_j(\theta^*)$ are disjoint, z also has a uniquely determined endpoint index. As the sign, endpoint index and θ are all that are needed to uniquely define $\tilde{e}_j^\pm(\theta)$ the map D must be injective. Therefore $D : \partial\Delta \to \partial\Delta$ is a bijection. ∎

It is important to realise that as this function D is defined using the sets formed by the **desym** algorithm it suffers the same dependency on the choice of points made in line 9 of **choose**. The important properties of this map in no way depend on this choice however, so we will either assume an arbitrary choice has been made or think of D as an equivalence class of all such maps. This decision will affect the definition of $f^\dagger$ in a similar fashion. The map D and the function $f^\dagger$ *are* dictated by the points $\alpha_1, \ldots, \alpha_n$, which we will assume to be fixed.

Definition 2.3.5. For each $f \in \mathcal{S}_n$ we define $f^\dagger : \partial\Delta \to \partial\Delta$ by

$$f^\dagger(z) = f(D^{-1}(z))$$

for each $z \in \partial\Delta$, where D is the bijection in Lemma 2.3.4.

In Figure 2.1 we see the steps in a desymmetrization of the three "hump" ramp function on $[-\pi.\pi]$. The top graph is the function being desymmetrized. The middle two show how far both the original function and its desymmetrization have "grown" before the first collision. The bottom two graphs are the same two functions after the second collision.

Proposition 2.3.6. *Let $f : C_r \to [0,1]$ be a continuous function. Then, whenever $f_\theta(re^{i\theta})$ exists,*

$$f^\dagger_\theta(re^{i\theta}) = f_\theta(re^{i\theta})$$

provided θ is not one of the θ_k. If $\theta = \theta_k$ then $f^\dagger$ has one-sided derivatives with respect to θ at $re^{i\theta}$ which are negatives of one another.

We can extend the definition of desymmetrization to complex valued function in a similar manner to that described for symmetrization. That is, the value of a function desymmetrized about α at a point $re^{i\theta}$ is $f^\dagger(re^{i\theta})$, where f is considered as a function of θ on C_r. A desymmetrization of a complex variable has the following useful property.

Proposition 2.3.7. *Let f be an admissible function on the annulus $\mathcal{A}(r_1, r_2)$. Then $f^\dagger$ is admissible on $\mathcal{A}(r_1, r_2)$ and*

$$I_{\mathcal{A}(r_1,r_2)}\left[f^\dagger\right] = I_{\mathcal{A}(r_1,r_2)}\left[f\right].$$

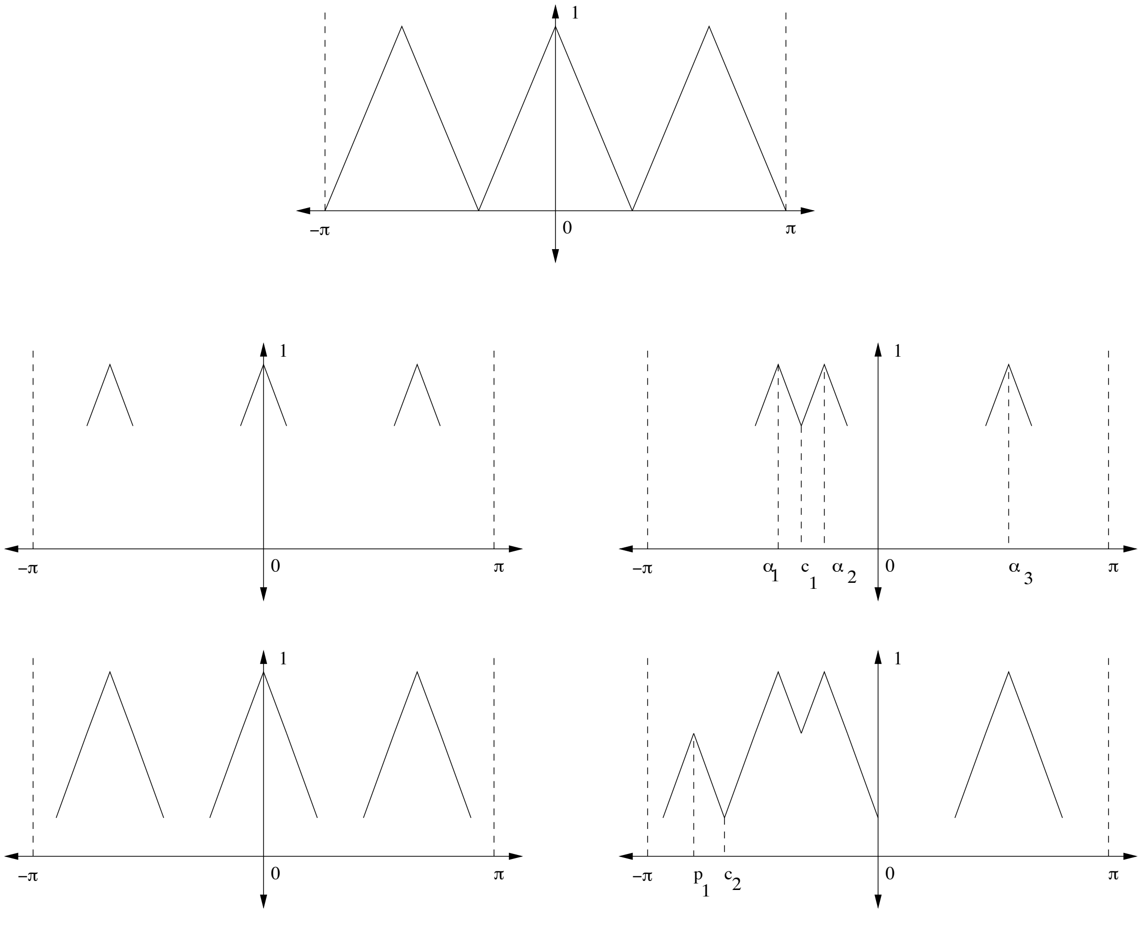

Figure 2.1: An Example Desymmetrization

Chapter 3

Proof of Major Results

Over the last two chapters we have covered a wide variety of ideas from the field of geometric function theory. In this chapter all we will bring together these topics and use them to prove the theorems of Dubinin and Baernstein. It is interesting to note the difference in style of both the proofs. In Dubinin's theorem we have assumed the radially slit domains are simply connected and so are able to take advantage of the Riemann mapping theorem and related results discussed in Section 1.3. This, along with the theory of the Dirichlet integral, gives us a very beautiful way of demonstrating the extremal property of harmonic measure on an evenly spaced, radially slit domain. In constrast, Baernstein's theorem uses a more "hands on" approach to analysing the properties of the harmonic measures. This is part due to the fact that Baernstein's theorem looks at a stronger inequality than Dubinin. Also, although restricted to three slits or less, Baernstein's radially slit domains are of a more general nature than Dubinin's, thus a closer inspection of the harmonic measures is warranted.

The proofs are presented here in two sections which can be read independently of each other. The section covering Baernstein's theorem is broken up into a subsection presenting some preliminary results and another containing the proof itself.

3.1 Proof of Dubinin's Theorem

In this section we will use the notation described in the opening chapter. To revise, n is a fixed positive integer, $K = [a, 1]$, while

$$\alpha = (\alpha_1, \ldots, \alpha_n) \quad , \quad \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$$

denote the arbitrary and symmetric positions of the slits on the boundary of the unit disk Δ respectively. These slits are the sets

$$S = \bigcup_{j=1}^{n} e^{i\alpha_j} K \quad , \quad \tilde{S} = \bigcup_{j=1}^{n} e^{i\tilde{\alpha}_j} K.$$

The domains formed by removing the slits in these two cases are Ω and $\tilde{\Omega}$ respectively. In the statement of Theorem 1.1.3 we are investigating $\omega(0, \partial\Delta, \cdot)$, the harmonic measure of the circular boundary of each domain. Dubinin originally stated his theorem in terms of the harmonic measure of the slits on each domain, and showed that if they were evenly spaced the harmonic measure at zero is maximised. We will let u and v denote the harmonic measures $\omega(z, \tilde{S}, \tilde{\Omega})$ and $\omega(z, S, \Omega)$ respectively, then Dubinin's original assertion was

Theorem 3.1.1. *For $u : \tilde{\Omega} \to [0, 1]$ and $v : \Omega \to [0, 1]$ just described we have*

$$v(0) \leq u(0)$$

with strict inequality unless $\tilde{\Omega}$ can be obtained from Ω via a rotation about the origin.

Since

$$S = \partial\Omega \setminus \partial\Delta \cup \{e^{i\alpha_j} : j = 1, \ldots, n\}$$

we can apply Proposition 1.2.13 and show that if $u(0) = \omega(0, \tilde{S}, \tilde{\Omega}) \geq \omega(0, S, \Omega) = v(0)$ then necessarily

$$1 - \omega(0, \partial\Delta, \tilde{\Omega}) \geq 1 - \omega(0, \partial\Delta, \Omega)$$

and so $\omega(0, \partial\Delta, \tilde{\Omega}) \leq \omega(0, \partial\Delta, \Omega)$. This shows that the theorem above implies Theorem 1.1.3. A similar argument in the reverse direction shows that they are equivalent statements. The reason Dubinin's theorem was introduced as Theorem 1.1.3 was to allow easy comparison between it and Baernstein's theorem (Theorem 1.1.4).

Presented below is a proof of Theorem 3.1.1 based on the one given in [4]. The first step is to shift the focus from the harmonic functions on the slit domains to the geometry of the domains themselves.

Recall in Section 1.3 we showed the value of the harmonic measure at zero is equal to the length of the image of the slits under a conformal map taking them to the unit circle. By (1.14) our aim is to prove that

$$|F(S)| \leq |\tilde{F}(\tilde{S})| \tag{3.1}$$

where $F : \Omega \to \Delta$ and $\tilde{F} : \tilde{\Omega} \to \Delta$ are the origin-fixing conformal maps discussed in Section 1.3. If it were possible, we would like to compose F^{-1} with $\tilde{F}$ and compare the lengths of $\tilde{S}$ with $F^{-1}(\tilde{F}(\tilde{S}))$. However, F and $\tilde{F}$ act on slits at different positions so such a composition of maps yields nothing. Dubinin's idea was to somehow make the slits coincide, so that this method of comparison could be used. A combination of symmetrization and desymmetrization allows him to do exactly that. The origin poses a slight problem when using these methods so we will side step it by considering radially slit annuli.

Using the annulus notation of Example 1.4.8 we form two families of doubly connected domains

$$\Omega(\delta) = \Omega \cap \mathcal{A}(\delta) \quad , \quad \tilde{\Omega}(\delta) = \tilde{\Omega} \cap \mathcal{A}(\delta)$$

for $\delta \in (0, a)$. By the mapping theorem for doubly connected domains we know there exists conformal maps $F_\delta : \Omega(\delta) \to \mathcal{A}(\varepsilon), \tilde{F}_\delta : \tilde{\Omega} \to \mathcal{A}(\tilde{\varepsilon})$ for each δ. The inner radii ε and $\tilde{\varepsilon}$ depend on δ as discussed in Section 1.3. The first result we need about these maps is the following

Lemma 3.1.2. *Using the notation above, as $\delta \to 0$ we have that*

$$F_\delta \to F \quad , \quad \tilde{F}_\delta \to \tilde{F}$$

uniformly on each compact subset of Ω and $\tilde{\Omega}$ respectively.

Sketch of proof. All that we will give here is an outline of the proof for the maps F_δ, the proof for the $\tilde{F}_\delta$ is covered by the same type of argument. Showing this type of convergence is generally dealt with using the *Carathéodory convergence theorem.* This theorem can be found in [9] and, for our purposes, states the following. Suppose we have a family of simply connected domains $\mathcal{D}_n$ such that

$$\bigcup_{n=1}^{\infty} \mathcal{D}_n = \mathcal{D} \neq \mathbb{C}$$

where $\mathcal{D}$ is also simply connected. Let f_n denote the conformal map taking $\mathcal{D}_n$ onto Δ. Then the f_n converge uniformly on compact subsets of $\mathcal{D}$ to the conformal map f that takes $\mathcal{D}$ to Δ. We would like to let $\mathcal{D}_n = \Omega(\frac{1}{n})$, that is take $\delta = \frac{1}{n}$ and apply the theorem verbatim. Unfortunately, these domains are not simply connected. We argue around this problem by lifting the maps F_δ to appropriate, simply connected, Riemann surfaces. Consider the Riemann surface created by taking an infinite number of copies of $\Omega(\frac{1}{n})$ and joining them in a corkscrew fashion along some branch cut. Call this surface $\hat{\Omega}(\frac{1}{n})$. This surface can be identified with a vertical infinite strip in $\mathbb{C}$ by taking a logarithm. There will be an infinite number of slits running up one side of this strip, the images of the slits of $\Omega(\frac{1}{n})$. Even with these slits this strip is simply connected and we can conformally map it onto the unit disk. Let Φ_n denote the compostion of the maps taking $\hat{\Omega}(\frac{1}{n})$ to the unit disk. This map is conformal. Let $\mathcal{A}(\varepsilon_n)$ (ε_n depending on n) be the image of $F_{\frac{1}{n}}$ on $\Omega(\frac{1}{n})$. The branch cut that was made to lift $\Omega(\frac{1}{n})$ to a Riemann surface will be mapped by $F_{\frac{1}{n}}$ onto some Jordan arc of $\mathcal{A}(\varepsilon_n)$. Using this arc as a join, we can form a corkscrew, $\hat{\mathcal{A}}(\varepsilon_n)$ from copies of $\mathcal{A}(\varepsilon_n)$. We then identify this Riemann surface with a vertical strip, this time without any slits, and conformally map it onto the disk. Let Ψ_n denote the conformal map taking $\hat{\mathcal{A}}(\varepsilon_n)$ to the disk. The union of the simply connected sets, $\hat{\Omega}(\frac{1}{n})$

will be $\hat{\Omega}$, the surface formed by joining copies of Ω. Also the union of the sets $\hat{\mathcal{A}}(\varepsilon_n)$ will converge to $\hat{\mathcal{A}}(\varepsilon_0)$ for some $\varepsilon \geq 0$. By the Carathéodory convergence theorem the maps Φ_n and Ψ_n will converge uniformly on compact subsets of $\hat{\Omega}$ and $\hat{\mathcal{A}}(\varepsilon_0)$ to conformal maps $\Phi : \hat{\Omega} \to \Delta$ and $\Psi : \hat{\mathcal{A}}(\varepsilon_0) \to \Delta$ respectively. By its construction, the projection of the maps $\Gamma_n = \Phi_n \circ (\Psi_n)^{-1}$ down to $\Omega(\frac{1}{n})$ will be $F_{\frac{1}{n}}$. The uniform convergence of the Φ_n and Ψ_n ensures that Γ_n converges to $\Gamma = \Phi \circ (\Psi)^{-1}$. This uniform convergence can be shown to carry through the projection and so $F_{\frac{1}{n}} \to G$ uniformly on compact subsets of Ω, where G is the restriction of Γ to Ω. Then G is a conformal map taking Ω to $\mathcal{A}(\varepsilon_0)$. As we have unform convergence of the $F_{\frac{1}{n}}$ we can show $\varepsilon_0 = 0$ proving the lemma. □

We can now, once again, shift the focus of the proof to showing that for all δ,

$$|F_\delta(S)| \leq |\tilde{F}_\delta(\tilde{S})|.$$

As we have just shown these maps converge to F and $\tilde{F}$ as $\delta \to 0$, if the above statement is true it will imply (3.1). Defined now is a series of functions that will eventually allow us to compare the lengths of the slit's images.

Let ν be the ramp function on the sector $\mathcal{D}_\delta(\delta, 0)$ as in Example 1.4.7. Using the Schwarz reflection principle we can repeatedly reflect this function in the lines $\arg z = \frac{j\pi}{n}$ for $j = 1, \ldots, 2n-1$ and extend ν to a function f defined on all of $\mathcal{A}(\delta)$. It is important to note that $f(re^{i\tilde{\alpha}_j}) = 1$ for all $r \in (\delta, 1]$. This means when we desymmetrize f about α, we have

$$S \subset (f^\dagger)^{-1}(\{1\}) = \{z : f^\dagger(z) = 1\}$$

since $f^\dagger(re^{i\alpha_j}) = f(re^{i\tilde{\alpha}_j})$ by property 4 of Lemma 2.3.2 regarding desymmetrizations. For convenience, we will let $f_1 = f^\dagger$. If we now conformally map $\Omega(\delta)$ onto $\mathcal{A}(\varepsilon)$ using F_δ, the slits S will be mapped onto $\partial\Delta$. We can define a function, f_2 on $\mathcal{A}(\varepsilon)$ by letting

$$f_2 = f_1 \circ F_\delta^{-1}.$$

Let $f_3 : \mathcal{A}(\varepsilon) \to [0, 1]$ be the n-fold symmetrization of f_2 as discussed in Section 2.1. We now want to define a function f_4 on $\tilde{\Omega}(\delta)$ by "pulling back" f_3 via the conformal map $\tilde{F}_\delta$. To be able to do this we require that $f_3(\tilde{F}_\delta(z))$ is defined for each $z \in \tilde{\Omega}$. That is, we want

$$\tilde{F}(\tilde{\Omega}) = \mathcal{A}(\tilde{\varepsilon}) \subseteq \mathcal{A}(\varepsilon)$$

The following lemma shows that $f_4 = f_3 \circ \tilde{F}_\delta$ can be properly defined and gives use a glimpse of how we can use desymmetrization to deduce results about our domains.

Lemma 3.1.3. *The inner radii ε and $\tilde{\varepsilon}$ satisfy*

$$\varepsilon \leq \tilde{\varepsilon}$$

for all δ, with strict inequality unless Ω can be obtained from $\tilde{\Omega}$ by a rotation about the origin.

Proof. Let $\tilde{h} : \tilde{\Omega} \to [0, 1]$ be the function defined by

$$\tilde{h}(z) = \omega(z, C_\delta, \tilde{\Omega}(\delta)).$$

That is, $\tilde{h}$ is the harmonic function on $\tilde{\Omega}$ which takes the value 1 on C_δ and 0 on the outer boundary consisting of the slits $\tilde{S}$ and $\partial\Delta$. Let $\tilde{h}^\dagger$ be a desymmetrization of $\tilde{h}$ about α. For comparison, let $h(z) = \omega(z, C_\delta, \Omega(\delta))$. We will consider all these functions continued onto the boundary of their domains if necessary. By virtue of the desymmetrization, $\tilde{h}^\dagger$ vanishes on the slits of $\Omega(\delta)$. Furthermore, $\tilde{h}^\dagger$ has the same values as h on C_δ and $\partial\Delta$. As h is harmonic, Proposition 1.4.5 tells us that

$$I_{\mathcal{A}(\delta)}[h] \leq I_{\mathcal{A}(\delta)}\left[\tilde{h}^\dagger\right]. \tag{3.2}$$

Here, the Dirichlet integral over all of $\mathcal{A}(\delta)$ is the same as over $\Omega(\delta)$ or $\tilde{\Omega}(\delta)$ since we can ignore the slits in the integration. By Proposition 2.3.7, we know that desymmetrization preserves the Dirichlet integral. Thus,

$$I_{\mathcal{A}(\delta)}\left[\tilde{h}^\dagger\right] = I_{\mathcal{A}(\delta)}\left[\tilde{h}\right].$$

Let $\tilde{g}(z) = \tilde{h}(\tilde{F}_\delta^{-1}(z))$ and $g(z) = h(F_\delta^{-1}(z))$ for z in $\tilde{\Omega}(\delta)$ and $\Omega(\delta)$ respectively. Then $\tilde{g}$ is harmonic on $\mathcal{A}(\tilde{\varepsilon})$ with boundary values zero on C_1 and one on $C_{\tilde{\varepsilon}}$. By the conformal invariance of the Dirichlet integral (Proposition 1.4.6) and the calculations of Example 1.4.8 we see that

$$I_{\mathcal{A}(\delta)}\left[\tilde{h}^\dagger\right] = I_{\mathcal{A}(\delta)}\left[\tilde{h}\right] = I_{\mathcal{A}(\tilde{\varepsilon})}[\tilde{g}] = 2\pi\left(\log\frac{1}{\tilde{\varepsilon}}\right)^{-1}.$$

Similarly,

$$I_{\mathcal{A}(\delta)}[h] = I_{\mathcal{A}(\bar{\varepsilon})}[g] = 2\pi\left(\log\frac{1}{\varepsilon}\right)^{-1}.$$

Substituting these values back into (3.2) gives

$$2\pi\left(\log\frac{1}{\varepsilon}\right)^{-1} \leq 2\pi\left(\log\frac{1}{\tilde{\varepsilon}}\right)^{-1}$$

which, through a simple manipulation, implies $\varepsilon \leq \tilde{\varepsilon}$, proving the first part of the lemma.

Consideration of the collision points of the desymmetrization of $\tilde{h}$ reveals that $\tilde{h}^\dagger$ cannot be C^1 in $\Omega(\delta)$ unless $\tilde{\Omega}(\delta)$ can be obtained from $\Omega(\delta)$ via a rotation about the origin. Thus, $\tilde{h}^\dagger$ will not be harmonic in the sectors $\mathcal{D}_\delta(\delta, \alpha_j)$. By Proposition 1.4.5 the inequality in (3.2) must then be strict, proving the lemma. □

At our disposal we now have the functions f, f_1, f_2, f_3 and f_4.

Lemma 3.1.4. *The functions f and f_4 just defined satisfy*

$$I_{\mathcal{D}}[f_4] \leq I_{\mathcal{D}}[f] \tag{3.3}$$

where $\mathcal{D} = \mathcal{D}_\delta(\delta, 0)$. The inequality is strict unless the points α_j are evenly spaced.

Proof. The function f was originally defined on $\mathcal{D}$ then extended to $\mathcal{A}(\delta)$ by using $2n$ copies of $\mathcal{D}$ as per the reflection principle. Using the invariance of the Dirichlet integral under desymmetrization and conformal transformation we see

$$\begin{aligned} I_{\mathcal{D}}[f] &= \frac{1}{2n} I_{\mathcal{A}(\delta)}[f] \\ &= \frac{1}{2n} I_{\mathcal{A}(\delta)}[f_1] \\ &= \frac{1}{2n} I_{\mathcal{A}(\varepsilon)}[f_2]. \end{aligned} \tag{3.4}$$

Recall $f_4 = f_3 \circ \tilde{F}_\delta$. The symmetry of f_3 and $\tilde{F}_\delta$ makes f_4 symmetric about the points $\tilde{\alpha}_j$ on each circle of $\mathcal{A}(\varepsilon)$. This, along with the conformal invariance of the Dirichlet integral again, gives

$$\begin{aligned} I_{\mathcal{D}}[f_4] &= \frac{1}{2n} I_{\mathcal{A}(\delta)}[f_4] \\ &= \frac{1}{2n} I_{\mathcal{A}(\tilde{\varepsilon})}[f_3]. \end{aligned} \tag{3.5}$$

We know from section 2.1 that the Dirichlet integral is decreased by an n-fold, symmetric rearrangement (see Proposition 2.1.9). f_3 is just such a rearrangement of f_2 so

$$I_{\mathcal{A}(\tilde{\varepsilon})}[f_3] \leq I_{\mathcal{A}(\varepsilon)}[f_2].$$

This inequality allows us to compare (3.4) and (3.5) giving

$$2n I_{\mathcal{D}}[f_4] \leq 2n I_{\mathcal{D}}[f]$$

from which we have shown the first part of the lemma. Under the assumption that the α_j are not evenly spaced, Lemma 3.1.3 implies $\mathcal{A}(\tilde{\varepsilon})$ is a proper subset of $\mathcal{A}(\epsilon)$. Let $B = \mathcal{A}(\epsilon) \setminus \mathcal{A}(\tilde{\varepsilon})$. B is then an open, non-empty subset of $\mathbb{C}$. By arguments given later in this section f_3 will have total variation $2n$ on B hence

$$\begin{aligned} I_B[f_3] &= \iint_B \left[(f_3)_\theta^2 + (f_3)_r^2\right] r\,dr\,d\theta \\ &\geq \iint_B (f_3)_\theta^2 \, r\,dr\,d\theta \\ &> 0 \end{aligned}$$

as $(f_3)_\theta$ is cannot be identically zero on B due to its total variation. Therefore,

$$\begin{aligned} I_{\mathcal{A}(\tilde{\varepsilon})}[f_3] &= I_{\mathcal{A}(\varepsilon)}[f_3] - I_B[f_3] \\ &\leq I_{\mathcal{A}(\varepsilon)}[f_2] - I_B[f_3] \\ &< I_{\mathcal{A}(\varepsilon)}[f_2] \end{aligned}$$

for the case when the α_j are not evenly spaced. This completes the proof. □

When restricted to $\mathcal{D} = \mathcal{D}(\delta, 0)\frac{\pi}{n}$, f is simply the ν function of Example 1.4.7. We saw in this example that of all admissible functions on $\mathcal{D}$ with boundary values like ν, ν had the smallest Dirichlet integral. The conclusion of Lemma 3.1.4 tells us that f_4 has a possibly smaller Dirichlet integral than f on $\mathcal{D}$. This implies that one of the following four must be true:

1. $f_4 = f$ on $\mathcal{D}$,
2. f_4 is not admissible on $\mathcal{D}$,
3. f_4 is not identically 0 on $\arg z = \frac{\pi}{n}$, or,
4. f_4 is not identically 1 on $\arg z = 0$.

Case 1 is ignored for if f_4 is the function f, equality would hold in 3.3 meaning that the α_j are evenly spaced. Case 2 is not true as f is admissible on $\mathcal{A}(\delta)$ hence so is f_1 by virtue of being a desymmetrization of f. As f_2 is the composition of f_1 with a conformal map it too must be admissible on $\mathcal{A}(\varepsilon)$. f_3 is then just a symmetrization of f_2 and so is admissible in $\mathcal{D}$. To see that case 3 does not hold requires a closer inspection of the processes involved in forming f_4.

Let γ be the line segment $\{z : \arg z = \frac{\pi}{n}\} \cap \mathcal{A}(\delta)$. Let γ_1 be the image of γ after desymmetrization about α. Then $f_1(\gamma_1) = \{0\}$ as $f(\gamma) = \{0\}$ by definition. Also, γ_1 is a straight line in $\mathcal{A}(\delta)$ with one endpoint on C_δ and the other at $e^{i\phi}$ for some $\phi \notin \{\alpha_1, \ldots, \alpha_j\}$. Put $\gamma_2 = F_\delta(\gamma_1)$. As F_δ is conformal, γ_2 will be a Jordan arc in $\mathcal{A}(\epsilon)$. The endpoint of γ_1 on C_δ will get mapped to a point on C_ε, and the point on C_1 will stay on C_1. This means any circle, $C_\rho, \varepsilon < \rho < 1$, will intersect γ_2 in at least one point. Also, $f_2(\gamma_2) = (f_1 \circ F_\delta^{-1})(F_\delta(\gamma_1)) = \{0\}$. Thus, when we perform a n-fold symmetrization on f_2 to get f_3, $f_3(z) = 0$ for all $z \in \gamma_3 \overset{\text{def}}{=} {\gamma_2}^\sharp = \{z : \arg z = \frac{\pi}{n}\} \cap \mathcal{A}(\epsilon)$.

The symmetry in the domain $\tilde{\Omega}(\delta)$ induces symmetry in the conformal map $\tilde{F_\delta}$, mapping the line $\arg z = \frac{\pi}{n}$ in $\mathcal{A}(\delta)$ to $\arg z = \frac{\pi}{n}$ in $\mathcal{A}(\tilde{\varepsilon})$. Let $\gamma_4 = (\tilde{F_\delta})^{-1}(\gamma_3)$. Since $\mathcal{A}(\tilde{\varepsilon}) \subseteq \mathcal{A}(\epsilon)$, f_3 restricted to $\mathcal{A}(\tilde{\varepsilon})$ is still zero along all of $\arg z = \frac{\pi}{n}$ in $\mathcal{A}(\tilde{\varepsilon})$. Hence

$$\begin{aligned} f_4(\gamma_4) &= (f_3 \circ \tilde{F_\delta})((\tilde{F_\delta})^{-1}(\gamma_3)) \\ &= f_3(\gamma_3) = \{0\}, \end{aligned}$$

and so f_4 is identically zero along $\arg z = \frac{\pi}{n}$.

This leaves case 4 as the only possible conclusion — f_4 is not identically 1 on $\arg z = 0$. We now use a similar argument to the one just given to show that the only place f_4 cannot be 1 is along the slits of $\tilde{\Omega}(\delta)$.

Recall that a is the distance from the centre of Δ to the endpoints of the slits. Let $\gamma = (\delta, \alpha) \subset \tilde{\Omega}(\delta)$, we are assuming, without loss of generality, that one of the slits lie on the positive real axis. Then γ is a line segment between the inner circle of $\tilde{\Omega}(\delta)$ and the innermost point of the slit on the real axis. Notice $f(\gamma) = \{1\}$. We desymmetrize f and its underlying domain about α. We can also assume one of the slits of $\Omega(\delta)$ lies on the positive real axis. As desymmetrization maps α_j to $\tilde{\alpha}_j$ we can fix it so that γ_1, the desymmetrization of γ, is in fact γ, and $f_1(\gamma_1) = \{1\}$. Conformally mapping γ_1 onto $\mathcal{A}(\varepsilon)$ by F_δ gives us a smooth arc. Call this arc γ_2. We know that F_δ maps the slits of $\Omega(\delta)$ onto the outer boundary of $\mathcal{A}(\varepsilon)$. This means the common endpoint of γ_1 and the slit lying on the positive real axis must get mapped to a point on C_1. Hence, γ_2 is a smooth arc in $\mathcal{A}(\varepsilon)$ with one endpoint on C_1 and the other on C_ε. Thus, by a similar argument to the previous case, f_3, the symmetrization of f_2, will be identically 1 along $\gamma_3 = \{z : \arg z = 0\} \cap \mathcal{A}(\tilde{\varepsilon})$. When $(\tilde{F}_\delta)^{-1}$ is used to map $\mathcal{A}(\tilde{\varepsilon})$ onto $\tilde{\Omega}(\delta)$, the endpoint of γ_3 with modulus 1 is mapped to the innermost point on the slit lying on the positive real axis. If we let $\gamma_4 = (\tilde{F}_\delta)^{-1}(\gamma_3)$ we see, once again by the symmetry of this conformal map, that $\gamma_4 = (\delta, a)$ and $f_4(\gamma_4) = \{1\}$. As f_4 cannot be identically 1 on $\arg z = 0$, we conclude that f_4 is not identically 1 on $[a, 1]$.

We are now ready to prove that $|F_\delta(S)| \leq |\tilde{F}_\delta(\tilde{S})|$.

First notice that the measure of $F_\delta(S)$ on $\partial\Delta$ is not affected by n-fold symmetrization. $F_\delta(S)$ is simply split into n equal length intervals about the $\tilde{\alpha}_j$ on $\partial\Delta$. By the symmetry of $\tilde{F}_\delta$ the n intervals of $\tilde{F}_\delta(\tilde{S})$ are also of equal length and evenly distributed about the $\tilde{\alpha}_j$. We know that $f_3(z) = 1$ for $z \in F_\delta(S)$ as f_1 is 1 on the slits of $\Omega(\delta)$. If $\tilde{F}_\delta(\tilde{S}) \subseteq F_\delta(S)$ then $f_4(z) = 1$ for all $z \in [a, 1]$ as $\tilde{F}_\delta([a, 1]) \subset \tilde{F}_\delta(\tilde{S})$ since $[a, 1] \subset \tilde{S}$ and $f_4 = f_3 \circ (\tilde{F}_\delta)^{-1}$. This contradicts f_4 not being identically 1 on $[a, 1]$. Hence $\tilde{F}_\delta(\tilde{S}) \subset F_\delta(S)$ on $\partial\Delta$ giving us the required inequality and so proving Theorem 3.1.1.

3.2 Proof of Baernstein's Theorem

The following section is a clarification of the proof given in [5] of Theorem 1.1.4. As in Baernstein's paper we give the proof for the three slit case and then explain how it can be easily modified for the two slit case. The single slit case is trivial as integral mean inequalities are invariant when rotated about the origin. Following Baernstein's method of proof, the major result is obtained via five lemmas that give us insight into the harmonic functions under scrutiny.

3.2.1 Prelude

The first three lemmas given here are fairly general results, the first of which is a way of calculating the laplacian of functions not unlike the $*$-functions of Section 2.2. In Baernstein's Lemma 1 of [5] there is a missing $\frac{1}{r^2}$ term which is included here. Although this error is made throughout his paper it makes no difference to the validity of the proof.

Lemma 3.2.1. *Let $f \in C^2(\mathcal{D}(a,b))$, $m = \frac{1}{2}(a+b), \delta = \frac{1}{2}(b-a)$, and $\lambda > 0$. Define g on $\mathcal{D}(0, \lambda^{-1}\delta)$ by*

$$g(re^{i\theta}) = \int_{m-\lambda\theta}^{m+\lambda\theta} f(re^{i\varphi})\, d\varphi.$$

Then the laplacian of g is given by

$$\int_{m-\lambda\theta}^{m+\lambda\theta} \Delta f(re^{i\varphi})\, d\varphi + \frac{(\lambda^2-1)}{r^2}\left[f_\theta(re^{i(m+\lambda\theta)}) - f_\theta(re^{i(m-\lambda\theta)})\right]. \tag{3.6}$$

Proof. The proof is just computation. We will need to work with the polar form of the laplacian, namely,

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Applying this to the definition of g gives,

$$\begin{aligned}\Delta \int_{m-\lambda\theta}^{m+\lambda\theta} f(re^{i\varphi})\, d\varphi = \int_{m-\lambda\theta}^{m+\lambda\theta} \frac{\partial^2}{\partial r^2} f(re^{i\theta})\, d\varphi + \int_{m-\lambda\theta}^{m+\lambda\theta} \frac{1}{r}\frac{\partial}{\partial r} f(re^{i\theta})\, d\varphi \\ + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\int_{m-\lambda\theta}^{m+\lambda\theta} f(re^{i\theta})\, d\varphi.\end{aligned} \tag{3.7}$$

We expand the third term in the sum above by manipulating derivatives of integrals and integrals of derivatives.

$$\begin{aligned}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\int_{m-\lambda\theta}^{m+\lambda\theta} f(re^{i\theta})\, d\varphi &= \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\lambda f(re^{i(m+\lambda\theta)} + \lambda f(re^{i(m-\lambda\theta)}\right] \\ &= \frac{1}{r^2}\left[\lambda^2 f_\theta(re^{i(m+\lambda\theta)}) - \lambda^2 f_\theta(re^{i(m-\lambda\theta)})\right] \\ &= \frac{1}{r^2}(\lambda^2-1)\left[f_\theta(re^{i(m+\lambda\theta)}) - f_\theta(re^{i(m-\lambda\theta)})\right] \\ &\quad + \frac{1}{r^2}\left[f_\theta(re^{i(m+\lambda\theta)}) - f_\theta(re^{i(m-\lambda\theta)})\right] \\ &= \frac{1}{r^2}(\lambda^2-1)\left[f_\theta(re^{i(m+\lambda\theta)}) - f_\theta(re^{i(m-\lambda\theta)})\right] \\ &\quad + \int_{m-\lambda\theta}^{m+\lambda\theta} \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} f(re^{i\varphi})\, d\varphi.\end{aligned} \tag{3.8}$$

Finally, by substituting (3.8) back into (3.7), we obtain,

$$\Delta g(re^{i\theta}) = \int_{m-\lambda\theta}^{m+\lambda\theta} \left(\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} \right) f(re^{i\varphi})\, d\varphi$$
$$+ \frac{1}{r^2}(\lambda^2 - 1) \left[f_\theta(re^{i(m+\lambda\theta)}) - f_\theta(re^{i(m-\lambda\theta)}) \right],$$

the required result. □

There will be need to look at the derivatives of functions near the boundary of domains. These derivatives will inevitably be one-sided so we introduce some notation used in [5] to make speaking of these derivatives more concise. Here's what we will call the *in* and *out* derivatives.

Definition 3.2.2.

$$f_\theta(re^{i\theta})_{\text{in}} \overset{\text{def}}{=} \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f(re^{i\theta}) - f(re^{i(\theta-\varepsilon)})}{\varepsilon},$$
$$f_\theta(re^{i\theta})_{\text{out}} \overset{\text{def}}{=} \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f(re^{i(\theta+\varepsilon)}) - f(re^{i\theta})}{\varepsilon}.$$

The "in" derivative of a function on $\mathcal{D}(a, b)$ can be used to give us the one-sided derivative pointing inwards from the boundary at $\arg z = b$. The "out" derivative can give us a similar derivative as we watch the value of the function as we move with increasing argument, from $\arg z = a$.

A couple of trivial observations regarding these definitions will come in useful later.

$$(-f_\theta)(re^{i\theta})_{\text{in}} = \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f(re^{i(\theta-\varepsilon)}) - f(re^{i\theta})}{\varepsilon},$$
$$(-f_\theta)(re^{i\theta})_{\text{out}} = \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f(re^{i\theta}) - f(re^{i(\theta+\varepsilon)})}{\varepsilon}.$$

The next lemma is basically saying the following: If the derivatives of a continuous function are "headed in the right direction" near a radial line taken out of a sector, and the function is superharmonic everywhere except this line, then the function is superharmonic everywhere. Precisely, we have,

Lemma 3.2.3. *Let $f \in C(\mathcal{D}(a, b))$ be superharmonic on $\mathcal{D}(a, b) \setminus \{z : \arg z = \theta_0\}$ for some fixed $a < \theta_0 < b$. Suppose that for all $r \in (0, 1)$,*

1. $f_\theta(re^{i\theta_0})_{in} \geq f_\theta(re^{i\theta_0})_{out}$,
2. $f_\theta(re^{i\theta_0})_{in} > -\infty$,

3. $f_\theta(re^{i\theta_0})_{out} < \infty$.

Then f is superharmonic in $\mathcal{D}(a,b)$.

Proof. Without loss of generality, assume $a < \theta_0 = 0 < b$. As superharmonicity is a local property we will prove Lemma 3.2.3 by showing that

$$f(x_0) \geq L(f; x_0, \delta)$$

for all $x_0 = x_0 e^{i\theta_0} \in (0,1), B(x_0,\delta) \subset \mathcal{D}(a,b)$. For a fixed x_0, δ let $\Delta_0 = B(x_0,\delta)$,

$$\Delta_0^+ = \Delta_0 \cap \{x + iy : y > 0\},$$

and h be the solution of the Dirichlet problem in Δ_0 with boundary conditions given by f. We define three new functions;

$$f_1(z) = f(z) - h(z),$$
$$f_2(z) = \frac{1}{2}(f_1(z) + f_1(\bar{z})),$$

and

$$f_3(z) = f_2(z) - t \arg z,$$

for $z \in \overline{\Delta_0}$, $t > 0$. It is clear from the super mean value property for superharmonic functions and the mean value property for harmonic functions that f_1 and f_2 are superharmonic. Clearly $\arg z$ is just the imaginary part of the analytic function $\log z$ and so harmonic. Hence f_3 is superharmonic on Δ_0^+ and continuous on the closure. Thus f_3 must achieve it minimum at some $z_0 \in \partial\Delta_0^+$. Now,

$$\begin{aligned}
(f_3)_\theta(r)_{\text{out}} &= \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{[f_2(re^{i\varepsilon}) - t\varepsilon] - [f_2(r)]}{\varepsilon} \\
&= \frac{1}{2}\limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f_1(re^{i\varepsilon}) - f_1(r)}{\varepsilon} - \frac{1}{2}\liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f_1(r) - f_1(re^{i-\varepsilon})}{\varepsilon} - t \\
&= \frac{1}{2}(f-h)_\theta(r)_{\text{out}} - \frac{1}{2}(f-h)_\theta(r)_{\text{in}} - t && (3.9)\\
&= \frac{1}{2}\left[f_\theta(r)_{\text{out}} - f_\theta(r)_{\text{in}}\right] + \frac{1}{2}\left[(-h)_\theta(r)_{\text{out}} - (-h)_\theta(r)_{\text{in}}\right] - t && (3.10)\\
&\geq -t, && (3.11)
\end{aligned}$$

for all $r \in \Delta_0 \cap (0,1)$. We get (3.9) by the definition of f_2 and the properties of the out and in derivative remarked on earlier. As h is harmonic in Δ_0 it has a continuous derivative over $\Delta_0 \cap (0,1)$ and so the out and in dervatives of h cancel in (3.10). Assumption 1 in the statement of the lemma then tells us $(f_3)_\theta(r)_{\text{out}} \geq -t$ and so $f_3(re^{i\theta})$ is decreasing as it moves away from $(0,1)$ with increasing θ. Hence f_3 cannot possibly take its minimum on $(0,1)$. This means $|z_0 - x_0| = \delta$. It follows that $f_2(z_0) = 0$ as f and h have the same boundary values. Since the maximum argument a point of Δ_0^+ can have is $\frac{\pi}{2}$ we see that

$$f_3(z) \geq f_3(z_0) = -t \arg z_0 \geq -t\frac{\pi}{2}$$

for $z \in \Delta_0^+$. We now see that f_2 must be non-negative in Δ_0^+ and hence $f_1(x) \geq 0$ for x on $(x_0 - \delta, x_0 + \delta)$ by the symmetry of f_2. By the definition of f_1 we then get

$$f(x_0) \geq h(x_0) = L(h; x_0, \delta) = L(f; x_0, \delta)$$

proving the lemma. □

The next lemma enables us to get a handle on what the derivative of a bounded, harmonic function is doing in a sector by examining its in and out derivatives at the boundary. It says that if the "flow" of the function is essentially non-positive at the boudaries then it is non-positive throughout the domain.

Lemma 3.2.4. *Suppose f is harmonic and bounded in $\mathcal{D}(a,b)$, continuous on $\overline{\mathcal{D}(a,b)} \setminus \{0, e^{ia}, e^{ib}\}$ and constant on $\{z : |z| = 1, a < \arg z < b\}$.If, for all $r \in (0,1)$,*

$$f_\theta(re^{ia})_{out} \leq 0$$
$$(-f)_\theta(re^{ib})_{in} \geq 0,$$

it follows that $f_\theta(z) \leq 0$ for all $z \in \mathcal{D}(a,b)$.

The proof given in [5] is long-winded and technical. The driving idea behind it is that as f is harmonic and bounded in $\mathcal{D}(a,b)$ is must have continuous derivatives of all orders. Hence,

$$\begin{aligned}\Delta(f_\theta) &= (f_\theta)_{rr} + \frac{1}{r}(f_\theta)_r + \frac{1}{r^2}(f_\theta)_{\theta\theta} \\ &= (f_{rr})_\theta + \frac{1}{r}(f_r)_\theta + \frac{1}{r^2}(f_{\theta\theta})_\theta \\ &= (\Delta f)_\theta \\ &= 0,\end{aligned}$$

and so f_θ is harmonic in $\mathcal{D}(a,b)$. We would therefore expect f_θ to satisfy the maximum priciple, that is, its value inside the sector should be less than its values on the boundary. Unfortunately, f_θ is not continuous on the boundary so we must resort to using the one-sided limit derivatives to see what is happening. The assumption that the function is constant on the circular boundary is so we can essentially ignore what the derivative is doing near there as it will be zero. With these ideas in mind the conclusion of the lemma seem quite natural. For a more rigourous demonstration of this lemma the reader is asked to look at Baernstein's proof.

Lemma 3.2.5. *Let $a < b < c$ such that $b-a \leq c-b$. Suppose f is subharmonic in $\mathcal{D}(a,c)$, harmonic in $\mathcal{D}(a,b) \cup \mathcal{D}(b,c)$, continuous on $\overline{\mathcal{D}(a,c)} \setminus \{0, e^{ia}, e^{ib}, e^{ic}\}$, and constant on*

$$\{z : |z| = 1, a < \arg z < b, b < \arg z < c\}.$$

Then, if the two-sided derivative $f_\theta(r_0 e^{ib})$ exists for some $r_0 \in (0,1)$, and one of

$$f(re^{ia}) \leq f(re^{ib}) \leq f(re^{ic}), \tag{3.12}$$

$$f(re^{ia}) \leq f(re^{ic}) \leq f(re^{ib}) \tag{3.13}$$

is true for all $r \in (0,1)$ we conclude that $f_\theta(r_0 e^{ib}) \geq 0$.

This somewhat complicated lemma is telling us that when the derivative, f_θ, exists on some point of $\{z : \arg z = b\}$, the function, f, will be increasing through this line provided the value on f is larger on $\{z : \arg z = b\}$ and $\{z : \arg z = c\}$ than it is on $\{z : \arg z = a\}$. The proof involves reflecting sectors about lines in $\mathcal{D}(a,c)$ so as to obtain a lower bound of zero for the difference

$$f(re^{i(b+\varphi)}) - f(re^{i(b-\varphi)}), \tag{3.14}$$

for all $r \in (0,1), \varphi > 0$. This would ensure the two-sided derivative, if it existed, would be non-negative.

Proof. We may assume $a = 0$ and so $c \geq 2b$. Let k be the smallest integer for which $c \leq 2b + kb$ and define, for $0 \leq j \leq k$,

$$c_j = c - jb.$$

Let $p = 2b$ and for the case when $k \geq 1$, define $m = \frac{1}{2}(b + c_{k-1})$. When m is defined, we know

$$m = \frac{1}{2}(b + (c - (k-1)b)) = \frac{1}{2}(2b + c_k) \leq 2b = p$$

as $c_k = c - kb \leq 2b$ by the definition of k. Also,

$$m = \frac{1}{2}(2b + c_k) \geq \frac{1}{2}(c_k + c_k) = c_k$$

as $c_k \leq 2b$. Hence $p \geq m \geq c_k$.

Now assume condition 1 in the statement of the lemma. That is, $f(r) \leq f(re^{ib}) \leq f(re^{ic})$ for $r \in (0,1)$. Now, for the case when $k = 0$, we have $c = 2b$ and so (3.14) harmonic for $\varphi \in \mathcal{D}(0,b)$ as f is harmonic in $\mathcal{D}(0,b) \cup \mathcal{D}(b,c)$. Also, by condition 1, the function in (3.14) is non-negative on the boundary of $\mathcal{D}(0,b)$ and so must be non-negative inside this domain proving the result for this case. Now take $k \geq 1$ and so $c_0 > 2b$. The function

$$f(re^{i(\frac{1}{2}c_0+\varphi)}) - f(re^{i(\frac{1}{2}c_0-\varphi)}) \tag{3.15}$$

is superharmonic in $\mathcal{D}(0, \frac{1}{2}c_0)$ as the left term is harmonic at each $\varphi \in \mathcal{D}(0, \frac{1}{2}c_0)$ and the right term is subharmonic in the same sector as we have assumed f to be subharmonic in $\mathcal{D}(0, b) \cup \mathcal{D}(b, c)$. Hence the difference is superharmonic. The function in (3.15) has non-negative boundary values hence is non-negative in all of $\mathcal{D}(0, \frac{1}{2}c_0)$. As $c_0 > 2b$, $re^{i(\frac{1}{2}c_0 - b)}$ is in $\mathcal{D}(0, \frac{1}{2}c_0)$ for all $0 < r < 1$, we get,

$$f(re^{ic_1}) = f(re^{i(c0-b)}) = f(re^{i(\frac{1}{2}c_0 + (\frac{1}{2}c_0 - b))}) \geq f(re^{ib}),$$

as,

$$f(re^{i(\frac{1}{2}c_0 + (\frac{1}{2}c_0 - b))}) \geq f(re^{i(\frac{1}{2}c_0 - (\frac{1}{2}c_0 - b))}),$$

since (3.15) is non-negative. As we know $\frac{1}{2}c_j \geq b$ for each $0 \leq j < k$ we can use an identical argument inductively on

$$f(re^{i(\frac{1}{2}c_j + \varphi)}) - f(re^{i(\frac{1}{2}c_j - \varphi)})$$

to prove that $f(re^{ic_j}) \geq f(re^{ib}), 0 \leq j \leq k$. Now consider the function

$$f(re^{i(m+\varphi)}) - f(re^{i(m-\varphi)})$$

for $0 < \varphi < m - b$. Using the argument above we can show that this function is superharmonic in $\mathcal{D}(0, m - b)$ with non-negative boundary values, hence substituting $\varphi = p - m < m - b$ gives $f(re^{ip}) \geq f(re^{ic_k})$. Finally, as $f(re^{ic_k}) \geq f(r)$, looking at the function in (3.14) and applying the same argument again, we show (3.14) is superharmonic for $0 < \varphi < b$. Thus we have a lower bound of zero to the derivative at re^{ib} if it exists.

Suppose now that condition 2 in the statement of the lemma is true. Let $q = c - \frac{1}{2}b$. We can show that the function

$$f(re^{i(\frac{1}{2}c - \varphi)}) - f(re^{i(\frac{1}{2}c + \varphi)})$$

is superharmonic in $\mathcal{D}(0, \frac{1}{2}c)$ by arguments used for the first case, and further that $f(re^{iq}) \geq f(re^{i(\frac{1}{2}b)})$. Let $m = \frac{1}{2}(b + c) \leq q$. If we reflect q in m we obtain the point $s = \frac{3}{2}b$ and we can obtain

$$f(re^{i\frac{3}{2}b}) \geq f(re^{iq}) \geq f(re^{i\frac{1}{2}b})$$

from which it follows (3.14) is superharmonic and we have proved the lemma. □

The next lemma is both the corner-stone of the proof of Baernstein's theorem for $n = 3$ and its stumbling block for proving it for larger numbers of slits. In Section 4.1 we will see computational evidence that the inequalities below may not hold for all $r \in (0, 1)$ when we increase the number of slits to four or more. It is also not clear how to get analogous inequalities for (3.16) without loss of generality with more than three slits.

Lemma 3.2.6. *Let $n = 3$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, satisfying*

$$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_1 + 2\pi$$
$$\alpha_2 - \alpha_1 \leq \alpha_3 - \alpha_2 \leq (\alpha_1 + 2\pi) - \alpha_3. \tag{3.16}$$

Let $v = h_\alpha$, the harmonic measure in Theorem 1.1.4. Then, for each $re^{i\theta} \in \mathcal{D}(0, \pi)$, we have

$$v(re^{i(m+\theta)}) \leq v(re^{i(m-\theta)}), \tag{3.17}$$

where $m = \frac{1}{2}(\alpha_1 + \alpha_2)$. Also,

$$v(re^{i\alpha_2}) \leq v(re^{i\alpha_1}) \leq v(re^{i\alpha_3})$$

for $0 < r < 1$.

Proof. The second inequality given in (3.16) tells us that

$$2\alpha_3 \leq \alpha_1 + \alpha_2 + 2\pi,$$

and so $\alpha_3 \leq m + \pi$. This means the points $re^{i\alpha_3}$ are in $\mathcal{D}(m - \pi, m + \pi)$ for $0 < r < 1$. Consider the function

$$h(re^{i\theta}) = v(re^{i(m+\theta)}) - v(re^{i(m-\theta)})$$

for $re^{i\theta} \in \mathcal{D}(0, \pi) \setminus e^{i\delta}K$ where K is the closed subset of $[0, 1]$ used to define Ω_α. The term $v(re^{i(m+\theta)})$ is subharmonic on points where $r \in K$ and $m+\theta = \alpha_3$ since $v(re^{i\alpha_3}) = 0$. Also, $v(re^{i(m-\theta)})$ is harmonic for all r and $m - \theta \neq \alpha_2$, that is, $\theta \neq \delta$, hence h is subharmonic in the given domain. Since h is the difference of two functions with the same boundary values it must be identically zero on $\partial\mathcal{D}(0, \pi)$ and therefore $h \leq 0$ inside by the maximum principle for subharmonic functions. Thus, we have proved (3.17), and by setting $\theta = \delta$ we obtain $v(re^{i\alpha_2}) \leq v(re^{i\alpha_1})$. An identical argument using $m = \frac{1}{2}(\alpha_3 + (\alpha_1 + 2\pi))$ shows that $v(re^{i\alpha_1}) \leq v(re^{i\alpha_3})$ completing the proof of the lemma. □

3.2.2 ...and Proof

To both refresh our memory and fix some notation that will be used in the proof of Theorem 1.1.4, we restate the theorem here using the $*$-function notation of Section 2.2. As in the introduction to the first chapter we ask that $K \subseteq [0, 1]$ be a closed set. For α and $\tilde{\alpha}$ respresenting an arbitrary and evenly spaced slit arrangement respectively, we form the radially slit domains $\Omega = \Omega_\alpha$ and $\tilde{\Omega} = \Omega_{\tilde{\alpha}}$. We use n to denote the number of slits in both domains. The harmonic measures of the circular part of these domains have the following relationship.

Theorem 3.2.7. *Let $n \leq 3$ and*

$$u(z) = \omega(z, \partial\Delta, \tilde{\Omega}) \quad , \quad v(z) = \omega(z, \partial\Delta, \Omega)$$

for $z \in \overline{\Delta}$. *Then, for all* $r \in (0,1), \theta \in [-\pi, \pi]$,

$$u^*(re^{i\theta}) \le v^*(re^{i\theta}).$$

In Section 2.2 we proved the equivalence of inequalities between integral means and inequalities relating $*$-functions. Thus, the above formulation of Baernstein's theorem is equivalent to that given in the introduction. The proof given in [5], which we shall follow closely, is quite computational and messy. The key idea behind all this computation, however, is quite straightforward. It is clear that we want to compare the $*$-functions of each harmonic measure. Unfortunately this is not such an easy task as the definition of a $*$-function involves taking a supremum of an integral over all sets of a given length. The symmetry of the domain $\tilde{\Omega}$ induces a symmetry in the harmonic measure u and hence in u^* as we will see. Exploiting this symmetry we can find exactly for which sets the integral defining the $*$-function of u takes its maximum. Then, using a construction inspired by desymmetrization, we will be able to compare u^* and v^*. As discussed in the introduction to this section we shall only give a proof of the three slit case and discuss modifications to prove the two slit case.

We can assume that the slit arrangement, α, satisfies

$$0 \le \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \stackrel{\text{def}}{=} \alpha_1 + 2\pi$$

and

$$\alpha_2 - \alpha_1 \le \alpha_3 - \alpha_2 \le \alpha_4 - \alpha_3. \tag{3.18}$$

By possibly reflecting the domain Ω and relabelling the α_j we can always satisfy this assumption without loss of generality. We will further assume that the inequalities between the length of the intervals in (3.18) are strict. If we have equality between all three intervals there is nothing to prove as Ω is simply a rotation of $\tilde{\Omega}$. If one of the inequalties is strict and the other is equality the proof can be easily modified to take this into account.

Let $\tilde{m}_j = \frac{1}{2}(\tilde{\alpha}_j + \tilde{\alpha}_{j+1})$ be the midpoints of the intervals $(\tilde{\alpha}_j, \tilde{\alpha}_{j+1})$ for $j = 1, 2, 3$. Let $I(c, \delta)$ denote the closed interval on $\partial\Delta$ (identified with $[-\pi, \pi]$) with centre c and length 2δ. Define a family of sets $\tilde{E}(\theta)$, for $0 \le \theta \le \frac{\pi}{3}$ by

$$\tilde{E}(\theta) = \bigcup_{j=1}^{3} I(\tilde{m}_j, \theta).$$

The E notation is used to draw attention to the similarities of these sets to the E sets in the desymmetrization section. We can picture these sets growing about the midpoints of the intervals $(\tilde{\alpha}_j, \tilde{\alpha}_{j+1})$, closing down on the $\tilde{\alpha}_j$ as they fill $\partial\Delta$. For $re^{i\theta} \in \mathcal{D}(0, \frac{\pi}{3})$ we define

$$u_1(re^{i\theta}) = \int_{\tilde{E}(\theta)} u(re^{i\varphi})\, d\varphi.$$

Writing out the integral explicitly we see

$$u_1(re^{i\theta}) = \sum_{j=1}^{3} \int_{\tilde{m}_j-\theta}^{\tilde{m}_j+\theta} u(re^{i\varphi})\, d\varphi.$$

Applying Lemma 3.2.1 to each summand above shows,

$$\begin{aligned}\Delta \int_{\tilde{m}_j-\theta}^{\tilde{m}_j+\theta} u(re^{i\varphi})\, d\varphi &= 0 + \frac{(1^2-1)}{r^2}\left[u_\theta(re^{i\tilde{m}_j+\theta} + u_\theta(re^{i\tilde{m}_j-\theta}\right] \\ &= 0\end{aligned}$$

for $j = 1, 2, 3$, since u is harmonic. Thus u_1 is harmonic as $\Delta u_1 = 0$.

The following lemma shows us the reason for constructing this function u_1.

Lemma 3.2.8. *Let u_1 be the function just defined. Then, for all $re^{i\theta} \in \mathcal{D}(0, \frac{\pi}{3})$, we have*

$$u_1(re^{i\theta}) = u^*(re^{i3\theta})$$

Proof. Let $K^3 = \{x^3 : x \in K\}$, and let $u_0(z) = \omega(z, \partial\Delta, \Delta \setminus K^3)$, the harmonic measure of $\partial\Delta$ with respect to the single slit domain $\Delta \setminus K^3$. As $z \mapsto z^3$ is a conformal map of the sector $\mathcal{D}(0, \frac{\pi}{3})$ to $\Delta \setminus [0, 1]$ we see that $u(z) = u_0(z^3)$ for $z \in \mathcal{D}(0, \frac{\pi}{3})$. We now calculate the in and out derivatives for the function $f(re^{i\theta}) = u_0(re^{i(\pi-\theta)})$ defined in the sector $\mathcal{D}(0, \pi)$.

$$\begin{aligned}f_\theta(r)_{\text{out}} &= \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{f(re^{i\varepsilon}) - f(r)}{\varepsilon} \\ &= \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{u_0(re^{i(\pi-\varepsilon)}) - u_0(re^{i\pi})}{\varepsilon} \\ &= (u_0)_\theta(re^{i\pi})_{\text{in}}.\end{aligned}$$

However, u_0 is harmonic along the negative real axis and so the in derivative there is just $(u_0)_\theta$. u_0 is also symmetric about the negative real axis by the Schwarz reflection principle, hence the derivative with respect to θ is zero there, showing $f_\theta(r)_{\text{out}} = 0$ for each $r \in (0, 1)$. Now

$$(-f)_\theta(re^{i\pi})_{\text{in}} = \limsup_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{u_0(re^{i\varepsilon}) - u_0(r)}{\varepsilon}.$$

If $r \notin K^3$ then the $(u_0)_\theta(r)$ exists and is zero by symmetry. If $r \in K^3$ then $u_0(r) = 0$. As u_0 is positive off K^3 by the maximum principle, we see $(-f)_\theta(()_{\text{in}} re^{i\pi}) \geq 0$. As f is constant on the boundary Lemma 3.2.4 implies

$$f_\theta(re^{i\theta}) = (u_0)_\theta(re^{i(\pi-\theta)}) \leq 0$$

for $re^{i\theta} \in \mathcal{D}(0, \pi)$. Hence $u_0(re^{i\theta})$ must be symmetrically descreasing about $\theta = \pi$ and so,

$$u_0{}^*(re^{i\theta}) = \int_{\pi-\theta}^{\pi+\theta} u_0(re^{i\varphi})\, d\varphi = \int_{\tilde{E}(\theta)} u_0(re^{i\varphi})\, d\varphi = u_1(re^{i\theta}),$$

as required. □

We now construct another function v_1 to compare to u_1. Let $m_j = \frac{1}{2}(\alpha_j + \alpha_{j+1})$ for $j = 1, 2, 3$. What we want to do is grow intervals around these midpoints so they close down upon the α_j. We do this by first growing an interval about m_3 until the space remaining between the endpoints of this interval and α_3 and α_4 is the same as the distance between α_2 and α_3. At this point we will start another interval about m_2 and enlarge it until the space between its endpoints and α_2 and α_3 is the same as the distance between α_1 and α_2. Finally we will define a third interval about m_1 and enlarge all three intervals until they have closed down upon the α_j. Here is a more rigorous construction.

Define ψ_1, ψ_2 and ψ_3 by the following equations.

$$\alpha_4 - (m_3 + \psi_1) = \alpha_3 - m_2$$
$$\alpha_4 - (m_3 + \psi_2) = \alpha_3 - (m_2 - \psi_3) = \alpha_2 - m_1.$$

ψ_1 is the amount the interval about m_3 is to enlarge before starting the second interval about m_2. ψ_2 and ψ_3 are then half the size of the intervals about m_3 and m_2 respectively when the final interval centred at m_1 starts. We denote the union of these intervals at a time θ in their growth by $E(\theta)$. Specifically,

$$E(\theta) = I(m_3, 3\theta),$$

for $0 \le \theta \le \theta_1$,

$$E(\theta) = I(m_3, \psi_1 + \tfrac{3}{2}(\theta - \theta_1)) \cup I(m_2, \tfrac{3}{2}(\theta - \theta_1)),$$

for $\theta_1 \le \theta \le \theta_2$, and

$$E(\theta) = I(m_3, \psi_2 + (\theta - \theta_2)) \cup I(m_2, \psi_3 + (\theta - \theta_2)) \cup I(m_1, (\theta - \theta_2)),$$

for $\theta_2 \le \theta\frac{\pi}{3}$. Each θ_j is the time at which the interval about the midpoint m_j begins to grow. We can define these times specifically by letting

$$\theta_1 = \tfrac{1}{3}\psi_1 \quad , \quad \theta_2 = \tfrac{\pi}{3} - (\alpha_2 - \alpha_1).$$

The factors 3 and $\frac{3}{2}$ in the definition of the $E(\theta)$ are to ensure that $|E(\theta)| = 6\theta = |\tilde{E}(\theta)|$ for each $\theta \in [0, \frac{\pi}{3}]$. We now define v_1 in a similar fashion to u_1, using the sets $E(\theta)$. For $re^{i\theta} \in \mathcal{D}(0, \frac{\pi}{3})$ let

$$v_1(re^{i\theta}) = \int_{E(\theta)} v(re^{i\varphi})\, d\varphi.$$

The key property of this function is its superharmoncity.

Lemma 3.2.9. *The function v_1 just defined is superharmonic in the sector $\mathcal{D}(0, \frac{\pi}{3})$.*

The proof of this lemma is quite involved and it is easy to get lost in the details. The main aim, however, is to show that $(\Delta v_1)(re^{i\theta}) \le 0$ for $re^{i\theta} \in \mathcal{B}$, where

$$\mathcal{B} \stackrel{\text{def}}{=} \mathcal{D}(0, \theta_1) \cup \mathcal{D}(\theta_1, \theta_2) \cup \mathcal{D}(\theta_2, \frac{\pi}{3}).$$

This would give us that v_1 is superharmonic in $\mathcal{D}(0, \frac{\pi}{3})$ except at the lines $\theta = \theta_j$. We then examine the in and out derivatives along these lines and apply some of the preceeding lemmas. The details can be found in Appendix A.

Given that v_1 is superharmonic, we are now in a position to prove that $u^* \leq v^*$. As

$$u_1(re^{i\theta}) = \sum_{j=1}^{3} \int_{m_j-\theta}^{m_j+\theta} u(re^{i\varphi})\, d\varphi,$$

an application of Lemma 3.2.1 reveals that u_1 is harmonic in $\mathcal{D}(0, \frac{\pi}{3})$. Hence,

$$w(re^{i\theta}) = v_1(re^{i\theta}) - u_1(re^{i\theta}) + \varepsilon\theta$$

is superharmonic in $S \stackrel{\text{def}}{=} \mathcal{D}(0, \frac{\pi}{3})$ for all $\varepsilon > 0$ and continuous on $S \setminus \{0\}$. Let $m = \inf\{w(z) : z \in \mathcal{S}\}$ and let $\{z_n\}$ be a sequence of points such that $w(z_n) \to m$ as $z_n \to z_0$. By the minimum principle for superharmonic functions we know $z_0 \in \partial S$. If $z_0 \in (0, 1]$, then

$$w(z_0) = v_1(z_0) - u_1(z_0) = 0$$

since $v_1(re^{i\theta})$ and $u_1(re^{i\theta})$ are clearly zero when $\theta = 0$. So for this case $m = 0$. If $|z_0| = 1$ and $0 < \arg z_0 \leq \frac{\pi}{3}$ we see that $w(z_0) = \varepsilon \arg z_0$ since $u = v = 1$ on $\partial\Delta$ and so $u_1 = v_1$ on the circular boundary of S. This means $m = w(z_0) = \varepsilon \arg z_0 > 0$, a contradiction since m is the infimum of w and $w(z_0) = 0$ for $z_0 \in (0, 1]$. Thus $z_0 \notin \{e^{i\theta} : 0 < \theta \leq \frac{\pi}{3}\}$. Now suppose $z_0 = r_0 e^{i\frac{\pi}{3}}$ for some $0 < r_0 < 1$. If $r_0 \in K$ we know that $u(r_0 e^{i\tilde{\alpha}_j}) = v(r_0 e^{i\alpha_j}) = 0$ for $j = 1, 2, 3$. Hence

$$(v_1)_\theta(z_0) = 2 \sum_{j=1}^{3} v(r_0 e^{i\alpha_j}) = 0$$

and similarly $(u_1)_\theta(z_0) = 0$. Using these results to caluculate $w_\theta(z_0)$ we see

$$w_\theta(z_0) = (v_1)_\theta(z_0) - (u_1)_\theta(z_0) + \varepsilon = \varepsilon > 0,$$

and so w is increasing as it approaches z_0 and so cannot have a minimum at $z_0 \in \{r_0 e^{i\frac{\pi}{3}} : r_0 \in K\}$. Now suppose $r_0 \notin K$ and let $J \subset [0, 1] \setminus K$ be the component containing r_0. By the definition of w, u_1 and v_1 we see that for all $r \in J$,

$$\begin{aligned} w(re^{i\frac{\pi}{3}}) &= \int_{-\pi}^{\pi} (v(re^{i\varphi}) - u(re^{i\varphi}))\, d\varphi + \varepsilon\frac{\pi}{3} \qquad (3.19) \\ &= 2\pi(v(0) - u(0)) + \varepsilon\frac{\pi}{3}, \end{aligned}$$

since u and v satisfy the mean value property and we are integrating around a simply connected component of the domain. This means $w(re^{i\frac{\pi}{3}})$ is constant for $r \in J$ and so $w(re^{i\frac{\pi}{3}}) = m$ on J. As w is continuous on $\overline{\mathcal{D}(0, \frac{\pi}{3})}$ the value of w at $r_1 e^{i\frac{\pi}{3}}$, for $r_1 = \inf K$, must also be m. By the previous argument, however, we know that this cannot be the case. Suppose now that $0 \notin J$. By equation (3.19) we can show that

$$w(re^{i\frac{\pi}{3}}) = A \log r + B$$

for some real constants A, B. Thus, w is extremal at the endpoints of J and by the continuity of w on the closure of its domain we see that w must take its minimum for some $r_1 \in K \cup \{1\}$. Again, by the previous case, we see that this cannot be true. This leaves us with the possibility that w takes its infimum at the origin. Again by (3.19), this would imply

$$m = 2\pi(v(0) - u(0)) + \varepsilon\frac{\pi}{3}$$

or $m = 0$ for the case that $0 \in K$ (that is, the slits meet at the origin). If $0 \notin K$ we use the argument for the case when $0 \in J$ to obtain $m = w(r_1 e^{i\frac{\pi}{3}})$ where $r_1 = \inf K$, a contradiction. A second, cursory, inspection of this entire argument will reveal that

$$0 \leq m = \inf w(z) \leq v_1(z) - u_1(z) + \varepsilon \arg z$$

for all $\varepsilon > 0, z \in \mathcal{D}(0, \frac{\pi}{3})$. Hence

$$v_1 \geq u_1. \tag{3.20}$$

Recall in the construction of v_1 the sets $E(\theta)$ had measure 6θ for each $\theta \in [0, \frac{\pi}{3}]$. Also,

$$\begin{aligned} v_1(re^{i\theta}) &= \int_{E(\theta)} v(re^{i\varphi})\, d\varphi \\ &\leq \sup_{|E|=6\theta} v(re^{i\varphi})\, d\varphi \\ &= v^*(re^{i3\theta}) \end{aligned}$$

by definition of v^*. Lemma 3.2.8 stated that $u_1(re^{i\theta}) = u^*(re^{i3\theta})$. This along with (3.20) proves for each $re^{i\theta} \in \mathcal{D}(0, \frac{\pi}{3})$.

$$u^*(re^{i3\theta}) = u_1(re^{i\theta}) \leq v_1(re^{i\theta}) \leq v^*(re^{i3\theta}).$$

Hence, for all $r \in (0, 1), \theta \in [-\pi, \pi]$ we have

$$u^*(re^{i\theta}) \leq v^*(re^{i\theta}),$$

proving the version of Baernstein's theorem in 3.2.7 for the case when $n = 3$ and $\alpha2 - \alpha_1 < \alpha_3 - \alpha_2 < \alpha_4 - \alpha_3$. For the case when $\alpha2 - \alpha_1 = \alpha_3 - \alpha_2$, say, the starting times θ_1 and θ_2 will be equal making the construction of v_1 and the proof it is superharmonic simpler in the details. Having only two slits also reduces the number of starting times to two making v_1 somewhat simpler. The only significant changes to the proof in either of these two cases occur when showing v_1 is superharmonic.

Chapter 4

Computations and Generalisations

This final chapter consists of results, experiments and other ideas that were encountered during the course of the writing of this thesis that may warrant further investigation. Of special interest is the reformulation of the theorems of Baernstein and Dubinin into a discrete setting. It is in this setting that Quine [16] was able to lend credence to Baernstein's theorem for the case when $n \geq 4$. The method described by Quine is adopted here and a vehicle for the computational investigation is given as code for the symbolic algebra package Maple in the appendix. Using this code we verify some of the results Quine obtained and extend them appropriately.

Before having read the contents of Quine's paper the abstract sparked my own investigation using finite element methods via the `PLTMG` package. Using this package I tried answering the question: What arrangement, if any, minimises the harmonic measure of slits of possibly different length in a radially slit domain? The results, on the whole, are regretably inconclusive, however the question is an interesting one.

The last part of this chapter is an overview of other methods and open questions relating to Baernstein's conjecture.

4.1 The Discrete Dirichlet Problem

The aim of this section is to provide a framework for the theorems of Dubinin and Baernstein in which we can computationally verify their conclusions for specific domains. The first problem we encounter is that there is no useful way of representing continuous functions or even $\mathbb{R}$ on a computer. We therefore need to "discretize" our domain and problem. The natural way to do is by using a grid of polar coordinates to represent points in a radially slit disk. Unfortunately, zero does not lend itself to this type of representation so our attention is turned to harmonic measure on a radially slit annulus. We now define precisely the type of domain at which we will be looking.

Definition 4.1.1. Let m and n be positive integers. The *discrete annulus* is the set

$$A_{m,n} \stackrel{\text{def}}{=} \{0, \ldots, m-1\} \times \mathbb{Z}_n$$

where $\mathbb{Z}_n$ is the integers taken modulo n. Each integer pair $(j,k) \in A_{m,n}$ corresponds to a complex number given by the map

$$(j,k) \mapsto e^{2\pi(-\frac{j}{m}+i\frac{k}{n})}$$

This map takes every discrete annulus $A_{m,n}$ onto the annulus, $\mathcal{A}(\varepsilon)$, of inner radius $\varepsilon = e^{-2\pi}$ and outer radius 1 in the complex plane.

Before we can speak of a Dirichlet problem on this annulus we need to specify what a harmonic function is in this setting. A function $u : A_{m,n} \to \mathbb{R}$ can be thought of as a matrix of values $u_{j,k}$ where $u_{j,k} = u(j,k)$. To simulate the mean value property in this discrete setting we define

$$L_D(u,(j,k)) = \frac{1}{4}\left(u_{j-1,k} + u_{j+1,k} + u_{j,k-1} + u_{j,k+1}\right)$$

for $1 \leq j \leq m-2$ and $k \in \mathbb{Z}_n$. A *discrete harmonic function* is then one that satisfies

$$u_{j,k} = L_D(u,(j,k)) \tag{4.1}$$

for all points $(j,k) \in \{1,\ldots,m-2\} \times \mathbb{Z}_n$. Put simply, the value of a discrete harmonic function at a point is the average of its values at the four adjacent points.

We place slits in $A_{m,n}$ by choosing $S_{\text{row}} \subset \{1,\ldots,m-1\}$, $S_{\text{col}} \subset \mathbb{Z}_n$ and highlighting points $(j,k) \in S_{\text{row}} \times S_{\text{col}}$. Here, the S_{row} corresponds to the set K and S_{col} the slit arrangement α in the real-valued case. We define a Dirichlet problem analogous to that on the slit disk by asking for a harmonic function, u on $A_{m,n}$ that satisfies

$$\begin{aligned} u_{0,k} &= 1, && \text{for } k \in \mathbb{Z}_n \\ u_{j,k} &= 0, && \text{for } (j,k) \in S_{\text{row}} \times S_{\text{col}} \\ u_{m-1,k} &= c, && \text{for } k \in \mathbb{Z}_n \\ \frac{1}{n}\sum_{k\in\mathbb{Z}_n} u_{m-2,k} &= c. \end{aligned} \tag{4.2}$$

for some constant c. The first two requirements above are clearly analogous to asking that the solution in the continuous Dirichlet problem approach 1 on $\partial\Delta$ and zero on the slits. The second two conditions are to compensate for the missing origin in the discrete case. We want our solution u to be a constant on the inner radius of $A_{m,n}$ as specified by the third condition. Also, the average of the solution u on the circle nearest to the inner boundary should equal this constant c. As the inner radius is small, $e^{-2\pi}$, we hope to use c as an approximation to $u(0)$ in the continuous case.

Providing justification for these discrete annuli approximations, Quine, in [16], proves the following.

Proposition 4.1.2. *Let Ω be the radially slit domain of Baernstein's theorem and for $\delta > 0$ let $\Omega(\delta) = \Omega \cap \mathcal{A}(\delta)$. Let*

$$g_\delta(z) = \omega(z, C_\delta, \Omega(\delta)) \quad , \quad h_\delta(z) = \omega(z, C_1, \Omega(\delta))$$

and

$$M(f; r) = \int_{-\pi}^{\pi} f(re^{i\theta})\, d\theta$$

for $\delta \leq r \leq 1$. Then, there exists a unique constant c_δ such that $M(cg_\delta + h_\delta; r)$ is constant as r varies between δ and $\inf K$. *Furthermore, as $\delta \to 0$, $c_\delta \to u(0)$, where $u(z) = \omega(z, \partial\Delta, \Omega)$.*

This shows that we can discretely approximate the harmonic measure of $\partial\Delta$ on $\Omega(\delta)$ which, in turn, approximates the harmonic measure of $\partial\Delta$ on Ω. This will give us the means of testing some special cases of the cases in Baernstein's theorem when $n \geq 4$. This type of testing lends itself naturally to computing.

The conditions in (4.1) and (4.2) can be formulated as a system of linear equations in $u_{j,k}$ and c. Solving this system gives the solution of the Dirichlet problem. The Maple code of Appendix B does exactly this. The main routine in this code is `numdp` which takes four arguments. The first two are the values of m and n defining the discrete annulus on which the problem is to be solved. The second two arguments are lists of positive integers defining the sets S_{row} and S_{col}. Upon receiving this input the procedure sets up the aforementioned system as a matrix equation and solves it, returning a matrix containing values for the function u. It should be noted that Maple solves this system exactly, returning rational numbers which can then be approximated by a truncated decimal expansion. The other procedures appearing in Appendix B are reasonably self-explanatory and are used to manipulate the results of `numdp`. It is worth pointing out that Quine indexes $u_{j,k}$ with $1 \leq j \leq m, 1 \leq k \leq n$ whereas `numdp` uses $0 \leq j \leq m-1, 0 \leq k \leq n-1$.

Quine solved the system of equations for u by an iterative method of approximations for 16 by 16 discrete annuli with four slits of varying shape and position. Using `numdp`, I was able to verify all of Quine's results as well as test his conjectures for five and six slit domains. Figure 4.1 and Figure 4.2 show graphically the tables Quine calculated in [16] for $S_{\text{row}} = \{3, 4\}, S_{\text{col}} = \{1, 3, 6, 11\}$ and $S_{\text{row}} = \{3, 4\}, S_{\text{col}} = \{1, 5, 9, 13\}$ respectively. Although these figures show qualitatively what the discrete harmonic measure are like it is not easy to infer values from the graph. For this the reader is asked to either look at Quine's paper or generate the numerical output using the Maple procedures provided.

Recall that in the proof of Baernstein's theorem we made use of Lemma 3.2.6. This lemma stated that for u, the harmonic measure of $\partial\Delta$ with respect to Ω, we had a monotonicity

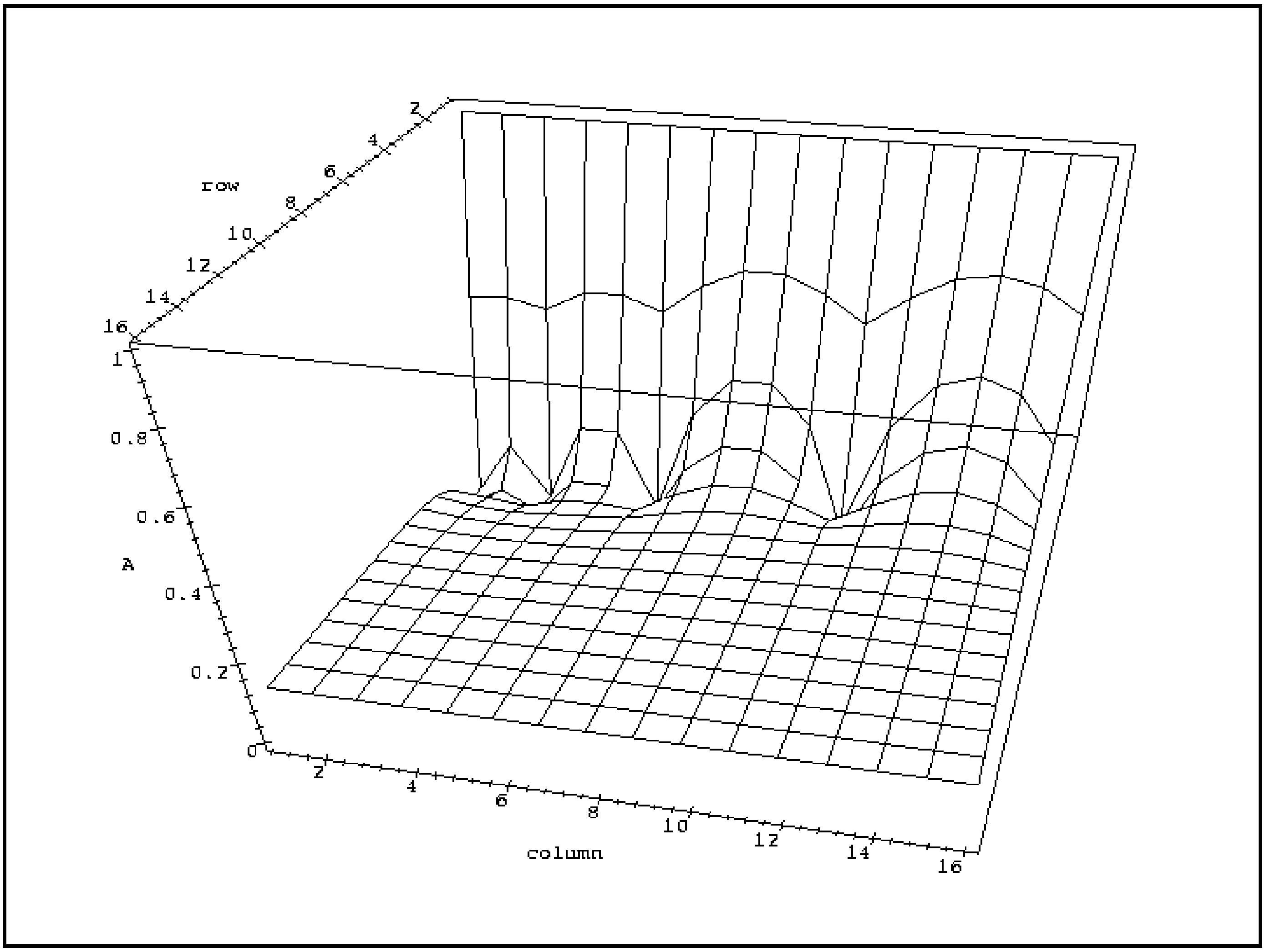

Figure 4.1: A non-symmetric slit arrangement

property for values of u along the radii on which the slits lie. Let $\alpha_1 = 1, \alpha_2 = 3, \alpha_3 = 6$ and $\alpha_4 = 13$ be the position of the four slits of Figure 4.1. We see then that

$$\alpha_2 - \alpha_1 < \alpha_3 - \alpha_2 < \alpha_4 - \alpha_3 < \alpha_1 + 2\pi - \alpha_4$$

which is comparable to the hypothesis of Lemma 3.2.6. Quine's tables show that $u_{5,1} = 0.102 < u_{5,6} = 0.103$ but $u_{6,1} = 0.136 > u_{6,6} = 0.134$. Thus, when we have four slits the approximate harmonic measure doesn't have the monotonicity property that would be required to sucessfully extend the proof beyond $n \leq 3$. This lack of monotonicity was also noticed using `numdp` for larger numbers of slits and varying arrangements.

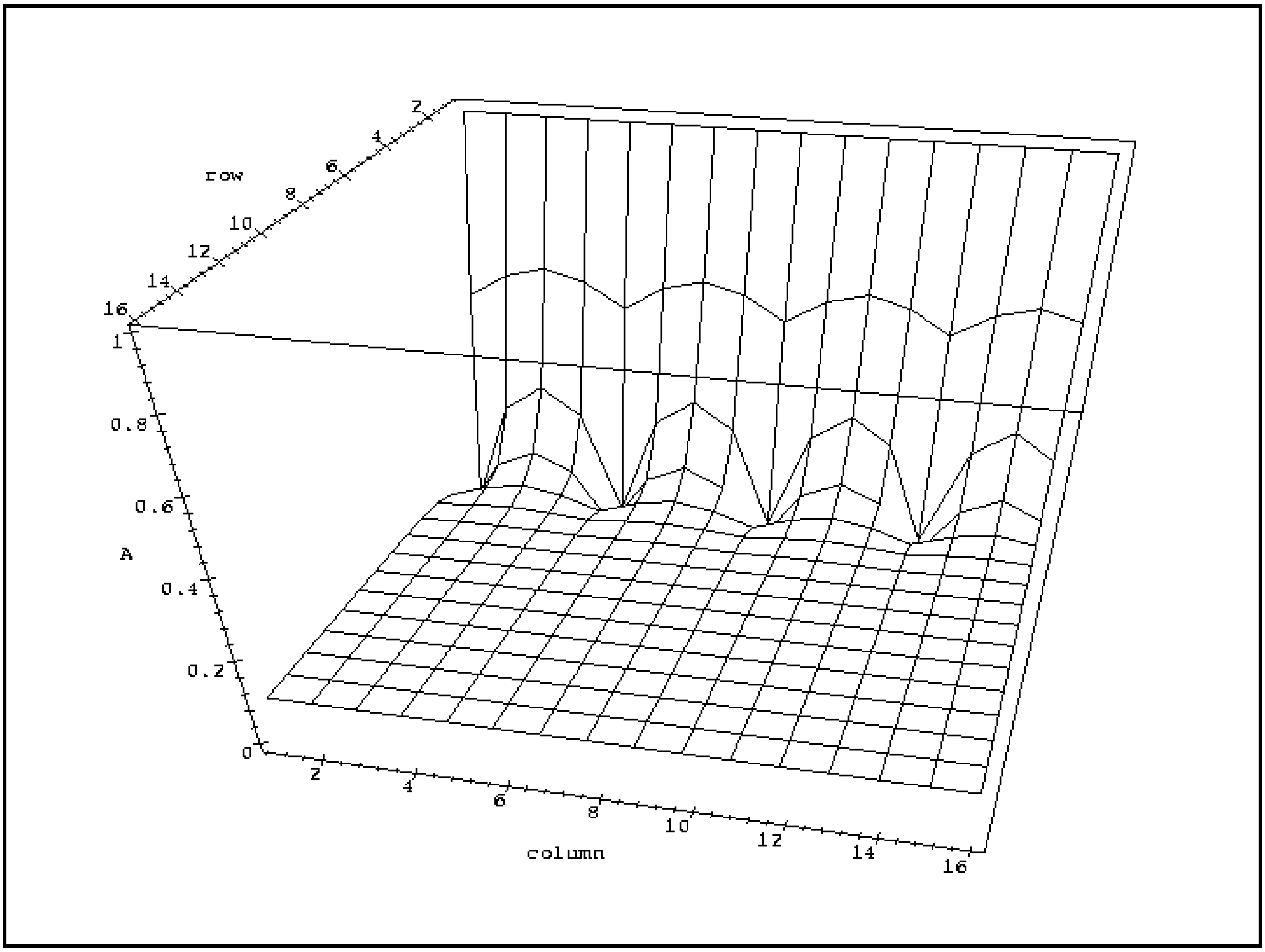

Figure 4.2: A symmetric slit arrangement

Although this bodes poorly for an extension of Baernstein's method of proof to slit domains of more than 3 slits there is some evidence that the conjecture may hold. For a given number of slits and a fixed set S_{row}, `numdp` was used to examine the function $u_{j,k}$ for various slit arrangements S_{col}. It was noticed that the value of the constant c was minimal when the slit arrangement S_{col} was evenly spaced. This would suggest that the $u(0) \leq v(0)$ inequality of Dubinin's theorem holds in the multiply connected case by the approximation arguments given earlier. Furthermore, we can also form analogues to Baernstein's $*$-functions in this discrete setting.

In Section 2.2 we saw that for $re^{i\theta} \in \mathcal{D}(0,\pi)$ the $*$-function of f could be expressed

$$f^*(re^{i\theta}) = \int_{-\theta}^{\theta} f^\sharp(re^{i\varphi})\, d\varphi$$

by Proposition 2.2.4. Motivated by this equivalence we set

$$u^*_{j,t} \stackrel{\text{def}}{=} \sum_{k=1}^{t} ({u_j}^\sharp)_k$$

where ${u_j}^\sharp$ is the decreasing rearrangement of the sequence $(u_{j,k})_{k=1}^n$. The above sum is then going to be maximum when compared to any other sum of t elements from $(u_{j,k})_{k=1}^n$. Quine, in [15], uses a definition of a discrete $*$-function based on finding this maximal sum. His definition compares readily to $*$-functions of a continuous variable, whereas the definition presented here is similar to $f^*(re^{i2\theta})$. The version presented here is slightly simpler to compute than Quine's. Using the Maple code in Appendix B I was able to implement this discrete $*$-function and test the inequality

$$u^* \leq v^*$$

where u and v are discrete version of the functions in Theorem 3.2.7. This test was performed for several four and five slit domains and it was found the inequality held. A proof of this inequality for all domains was recently brought to my attention and can be found in [15].

4.2 Investigations with PLTMG

This section discusses the use of a software package called PLTMG that was used to obtain some quantitative results for harmonic measures on simply connected domains. The aim here was to investigate what happens to Dubinin's theorem (version 3.1.1) when we loosen some constraints on the slits. In particular, what can be said about the arrangement of slits that maximises harmonic measure at zero when the slits are not all of the same length?

This software also made for an excellent visualisation tool. Figure 1.1 in Section 1.2 is an example of the type of graphical output PLTMG is capable of producing.

Consider a simply connected, radially slit domain with n slits of equal length. Dubinin's theorem tells us that the harmonic measure of the slits of this domain is maximised when these slits are evenly spaced. If we then shrunk one of these slits so it became a point on the boundary the maximal arrangement would now be the same as that for $n-1$ slits. When the shrinking slit is between these two extremes it seems natural that the maximal slit arrangement should be somewhere "between" the ones for n and $n-1$ slits. Using

PLTMG as a base, I was able to write code that allowed the relationship between differing slit length and these arrangements to be investigated.

PLTMG uses a finite element method to solve fairly arbitrary partial differential equations on many varied types of domain. We saw in Section 1.2 that harmonic functions are those which satisfy the partial differential equation

$$\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial^2 f}{\partial y^2}(x, y) = 0$$

for all points (x, y) in a given domain. Using the code in Appendix C.1, this partial differentiation, along with a description of the slit domain and boundary conditions is given to PLTMG which proceeds to approximate a solution. The methods used to find this approximation are discussed in [6]. A value at the origin of the harmonic measures of Theorem 1.1.3 can then be ascertained for varying domains. The accuracy of this approximation can be tested for specific domains by recalling equation (1.13) in Section 1.3. For a three slit domain $\tilde{\Omega}$ where the slits $\tilde{S}$ are of equal length and evenly spaced we have

$$\omega(0, \tilde{S}, \tilde{\Omega}) = \frac{2}{\pi} \cos^{-1} \left(\frac{2(0.5)^{\frac{3}{2}}}{(0.5)^3 + 1} \right) \approx 0.5673.$$

This domain was set up and a value calculated with PLTMG, which returned 0.566, a reasonable correspondence. This is a fairly nice example as the slit length is not extreme. For slit lengths near 0 and 1 the accuracy of PLTMG decreased substantially.

Let n denote a fixed number of slits, and denote by $K_1, \ldots, K_n$ intervals in $[0, 1]$ of the form $[a_n, 1]$. For a given slit arrangement $\alpha = (\alpha_1, \ldots, \alpha_n)$ we form the slits

$$S_\alpha = \bigcup_{j=1}^{n} e^{i\alpha_j} K_j$$

and the slit domain $\Omega_\alpha = \Delta \setminus S_\alpha$. Let u_α denote the harmonic measure $\omega(z, S_\alpha, \Omega_\alpha)$. We can now think of $u_\alpha(0)$ as a function of the slit positions $\alpha = (\alpha_1, \ldots, \alpha_n)$. Denote this function of α by $u_0(\alpha) : [0, 2\pi]^n \to [0, 1]$. Finding the arrangement which maximises the harmonic measure at zero is now a matter of maximising $u_0(\alpha)$. A "brute force" method of finding this maximum is adopted using the approximations to u_0 from PLTMG. This method involves partitioning the space $[0, 2\pi]$ and testing the value of u_0 at each point in this partition. By checking every possible arrangement on a fine enough grid a reasonably accurate maximal slit arrangement can be found. Fortran code using both the procedure in Appendix C and the PLTMG service procedures was written to implement this search on three slit domains. It was tested on a case where the slits had length 0.5, and a partition of $[0, 2\pi]$ into 50 points was used. It reported that a maximal arrangement with slits at $0, 0.66\pi$ and 1.33π which is close to what is expected. To keep computational overheads down the domains investigated only consisted of three slits and that one of these slits was fixed on the positive real axis. Furthermore, it was required these slits to satisfy

$$c|K_1| = |K_2| = |K_3|$$

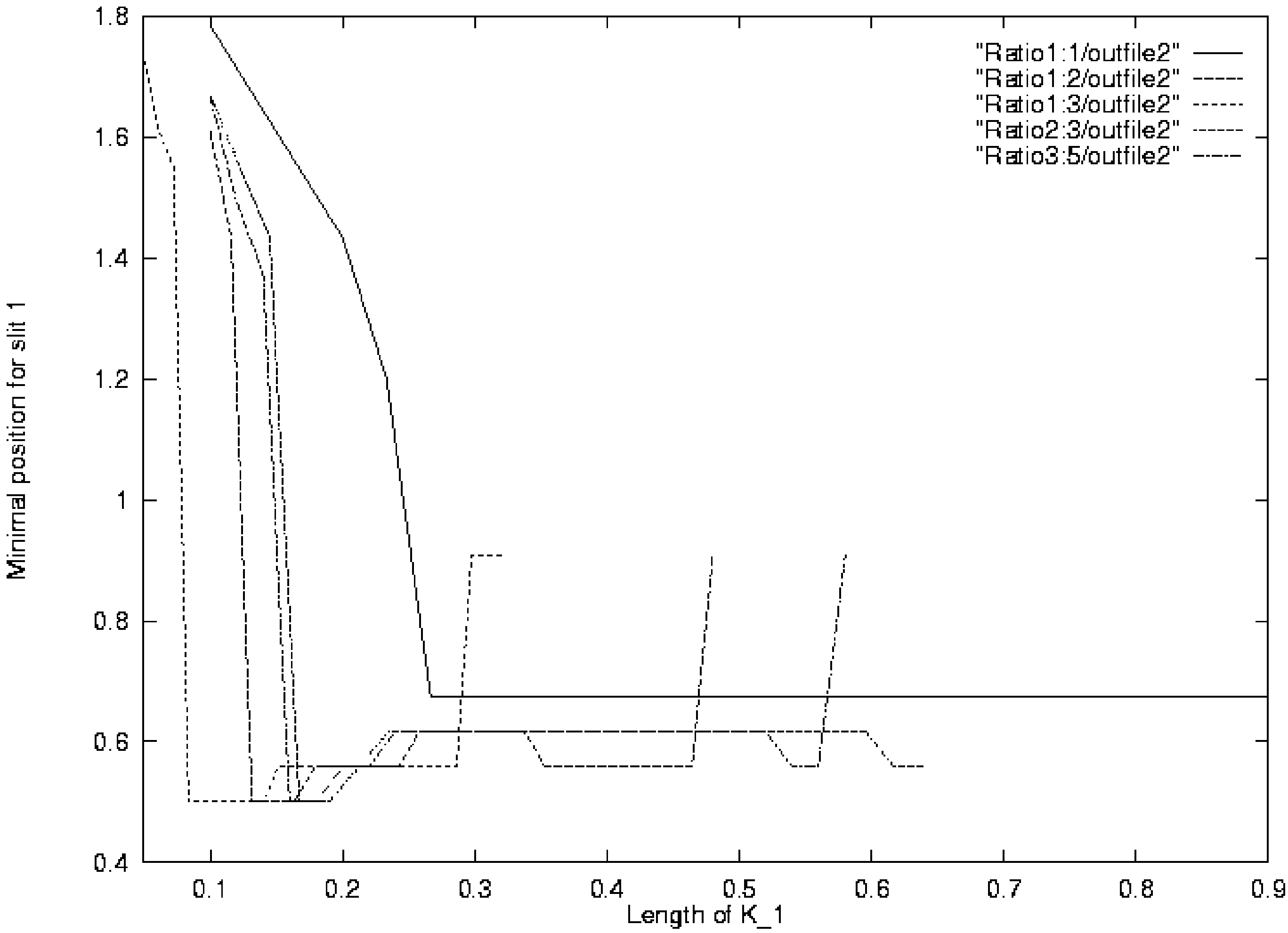

Figure 4.3: Minimal position for α_1 against length of K_1

for some $0 < c < 1$. It was hoped that for a given c the minimal arrangement would be independent on the length of K_1. The method used to check this hypothesis computationally was as follows. For a given c, let $|K_1|$ range through a discrete set of values in $(0, \frac{1}{c})$. For each slit K_1 fix K_2 and K_3 appropriately and use the "brute force" hunt for the maximal arrangement described previously. The Fortran implementation for this is also included in Appendix C.2. This code was tested for $c = 1, \frac{5}{3}, \frac{3}{2}, 2$ and 3, stepping the lengths through 25 values and partitioning $[0, 2\pi]$ into 25 intervals. Figure 4.3 and Figure 4.4 show the position for maximal harmonic measure of the first and second slit respectively as a function

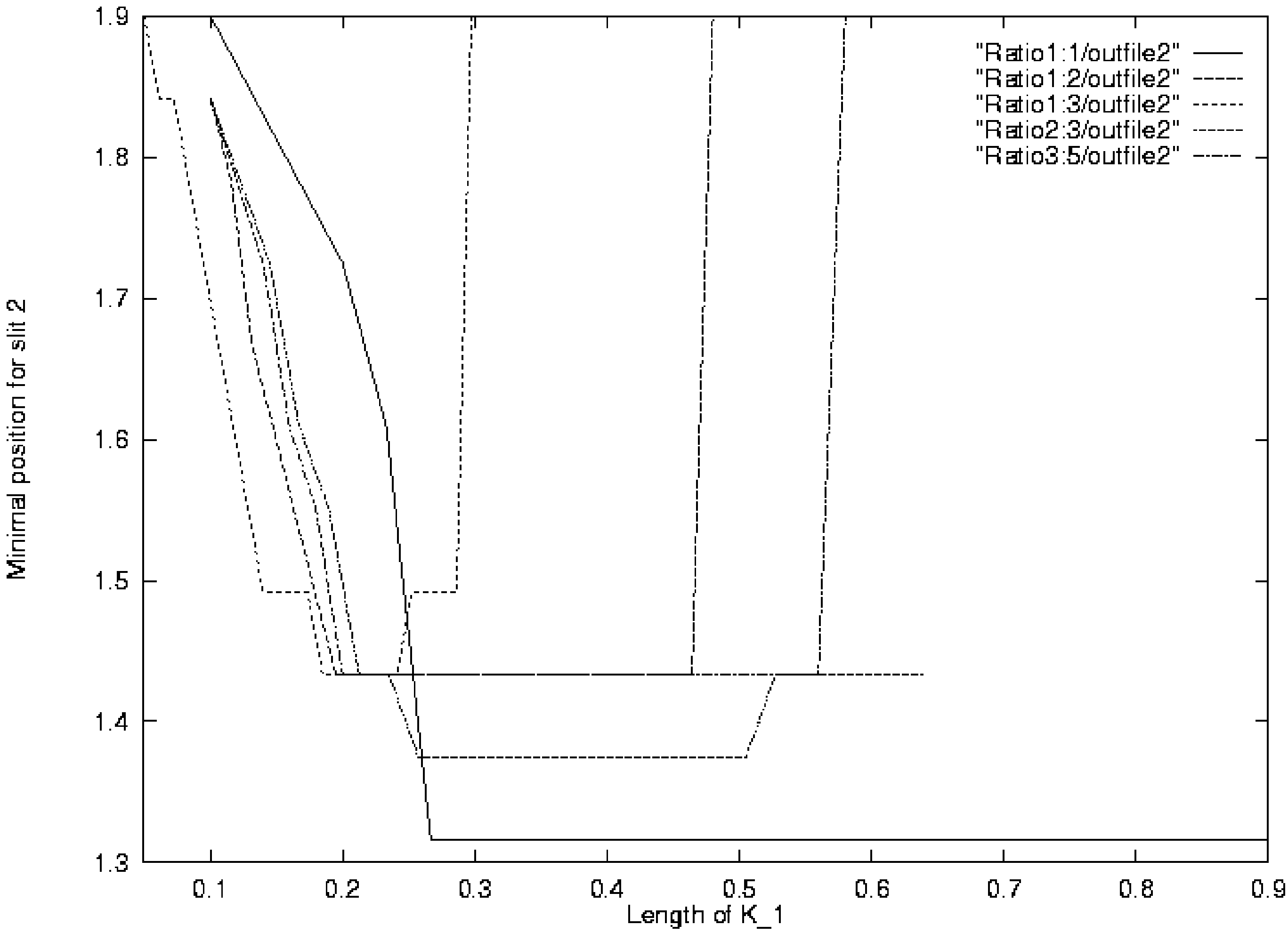

Figure 4.4: Minimal position for α_2 against length of K_1

of the length of K_1. The vertical scale is in multiples of π radians.

As can be seen in these figures, for small or large values of $|K_1|$ the position the program returns for the maximal arrangement is quite unrealistic. Near the middle of each graph we can see that the results are a little more stable although still fairly inconclusive as the positions of the slits are quantised by the partitioning of $[0, 2\pi]$. It is thought that with a finer partitioning these graphs may be of some interest, but this would have taken a great deal of computer time.

4.3 Conclusions

Throughout the course of this thesis we have touched on several topics in geometric function theory. As well as the more classical areas such as the theory of the Dirichlet integral and conformal maps, we have been witness to the power and elegance of the relatively new theories of symmetrization and desymmetrization. These ideas will no doubt be employed in future proofs of theorems in a similar vein to those of Dubinin and Baernstein.

It is hoped that this thesis has imparted a "feel" for some of the properties of harmonic functions and that the reader can see that the conjecture of Baernstein's should be true for the more general cases. The computational results of this final chapter lend credence to this gut feeling, at least in the case of four and five slit domains, by approximating harmonic measures and $*$-functions. However, they also point to the need for a new method of proof, as the crucial Lemma 3.2.6 in Baernstein's proof seems unlikely to be true for even four slit domains. It seems that some more investigation of $*$-functions and integral means may be needed before proceeding further. As mentioned in the introduction, Kakutani's theorem relates harmonic measure to hitting probabilities of Brownian motions. It is thought by Baernstein, amongst others, that this characterization of harmonic measure may be a useful on in coming up with a general proof.

The generalization of slit domains given in Section 4.2, I feel, is an interesting one. Although the computational methods used were somewhat crude, there seemed to be a definite trend in the relation between the maximal arrangements and the ratio of slits which may be worth pursuing.

Appendix A

Proof of Lemma 3.2.9

Proof of Lemma 3.2.9. Notice that v_1 is defined in terms of the integral of the C^2 function v. Therefore we can use Lemma 3.2.1 to calculate Δv_1. As v is, by definition, harmonic for $z \in \Delta \setminus \{z : \arg z = \alpha_j, j = 1, 2, 3\}$ the integral term in (3.6) for Δv_1 will be zero since $\Delta v = 0$. Therefore we have

$$\begin{aligned}(\Delta v_1)(re^{i\theta}) &= \frac{3^2 - 1}{r^2}\left[v_\theta(re^{i(m_3+3\theta)}) - v_\theta(re^{i(m_3-3\theta)})\right] \\ &= \frac{8}{r^2}(h_1)_\varphi(re^{i3\theta}),\end{aligned}$$

for $0 \le \theta \le \theta_1$, where

$$h_1(re^{i\varphi}) = v(re^{im_3+\varphi}) + v(re^{im_3-\varphi})$$

and the derivative with respect to φ is to done symbolically on h_1 before being evaluated at 3θ. For $\theta_1 \le \theta \le \theta_2$,

$$\begin{aligned}(\Delta v_1)(re^{i\theta}) = {} & \frac{(\frac{3}{2}^2 - 1)}{r^2}\left[v_\theta(re^{i(m_3+\psi_1+(\theta-\theta_1))}) - v_\theta(re^{i(m_3-\psi_1-(\theta-\theta_1))})\right] \\ & + \frac{(\frac{3}{2}^2 - 1)}{r^2}\left[v_\theta(re^{i(m_2+(\theta-\theta_1))}) - v_\theta(re^{i(m_2-(\theta-\theta_1))})\right] \\ = {} & \frac{5}{4r^2}(h_3)_\varphi(re^{i\frac{3}{2}(\theta-\theta_1)})\end{aligned}$$

where

$$\begin{aligned}h_3(re^{i\varphi}) &= h_1(re^{i(\psi_1+\varphi)}) + h_2(re^{i\varphi}), \\ h_2(re^{i\varphi}) &= v(re^{i(m_2+\varphi)}) + v(re^{i(m_2-\varphi)}).\end{aligned}$$

When $\theta_2 \le \theta \le \frac{\pi}{3}$, $v_1(re^{i\theta})$ is the integral over three separate intervals, all moving with "velocity" 1. When Δv_1 is calculated using Lemma 3.2.1 the constant 1 in front of the θ

term in each integral gets used for λ in equation (3.6) making the $\frac{\lambda^2-1}{r^2}$ term vanish. As mentioned earlier, the integral term in the expression is zero since v is harmonic, and so

$$(\Delta v_1)(re^{i\theta}) = 0$$

for $re^{i\theta} \in \mathcal{D}(\theta_2, \frac{\pi}{3})$. This means v_1 is superharmonic in $\mathcal{D}(\theta_2, \frac{\pi}{3})$.

To show v_1 is superharmonic in $\mathcal{D}(0, \theta_1)$ we will show that $(h_1)_\varphi(re^{i\varphi}) \leq 0$ for $r \in (0,1)$ and $0 < \varphi < \delta = \frac{1}{2}(\alpha_4 - \alpha_3)$. Looking at the definition of h_1 we see $h_1(re^{i\varphi})$ is harmonic in $\mathcal{D}(0, \delta)$. Furthermore, we know $v(z)$ is smooth at $\arg z = m_3$ and so $h_1(z)$ is smooth for $\arg z = 0$. Hence

$$(h_1)_\varphi(r)_{\text{out}} = (h_1)_\varphi(r) = v_\varphi(re^{im_3}) - v_\varphi(re^{im_3}) = 0.$$

It is sufficient then, according to Lemma 3.2.4, to show that $(-h_1)_\varphi(re^{i\delta})_{\text{in}} \geq 0$ to show $(h_1)_\varphi \leq 0$. We know

$$(-h_1)_\varphi(re^{i\delta})_{\text{in}} = \limsup_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{h_1(re^{i(\delta-\varepsilon)}) - h_1(re^{i\delta})}{\varepsilon}.$$

Noticing that $m_3 + \delta = \alpha_4$ and $m_3 - \delta = \alpha_3$ we get,

$$\begin{aligned} h_1(re^{i(\delta-\varepsilon)}) - h_1(re^{i\delta}) &= v(re^{i(m_3+\delta-\varepsilon)}) + v(re^{i(m_3-\delta+\varepsilon)}) - v(re^{i(m_3+\delta)}) - v(re^{i(m_3-\delta)}) \\ &= v(re^{i(\alpha_4-\varepsilon)}) + v(re^{i(\alpha_3+\varepsilon)}) - v(re^{i\alpha_4}) - v(re^{i\alpha_3}) \qquad \text{(A.1)} \end{aligned}$$

If $r \in K$ then (A.1) is positive since $v(re^{i\alpha_4}) = v(re^{i\alpha_3}) = 0$ and $v \geq 0$ everywhere. For $r \notin K$, v is smooth near each of the $re^{i\alpha_j}$, hence h_1 is smooth near $re^{i\delta}$. So $(-h_1)_\varphi(re^{i\delta})_{\text{in}} = -(h_1)_\varphi(re^{i\delta})$. Since, by our assumption, we have $\alpha_3 - \alpha_2 < \alpha_4 - \alpha_3$, so we can apply Lemma 3.2.6 to obtain

$$v(re^{i\alpha_2}) \leq v(re^{i\alpha_4}) \leq v(re^{i\alpha_3}).$$

Using this, we apply Lemma 3.2.5 for u in the sector $\mathcal{D}(\alpha_2, \alpha_3)$ and see that $v_\varphi(re^{i\alpha_3}) \geq 0$. Now consider the reflection of v through the bisecting radius, $\arg z = m_2 + \pi$, of $\mathcal{D}(\alpha_3, \alpha_2 + 2\pi)$. A similar argument to that for $v_\varphi(re^{i\alpha_3})$ shows that $v_\varphi(re^{i\alpha_4}) \leq 0$. Hence,

$$(h_1)_\varphi(re^{i\delta}) = v_\varphi(re^{i\alpha_4}) - v_\varphi(re^{i\alpha_3}) \leq 0 \qquad \text{(A.2)}$$

and therefore $(h_1)_\varphi \leq 0$ in $\mathcal{D}(0, \delta)$ proving v_1 is superharmonic in $\mathcal{D}(0, \theta_1)$.

Our next step is to show that $\Delta v_1 \leq 0$ in $\mathcal{D}(\theta_1, \theta_2)$. By the above computation of Δv_1 in this range, it is sufficient to show that $(h_3)_\varphi \leq 0$ in $\mathcal{D}(0, \delta)$ where δ is now $\frac{1}{2}(\alpha_3 - \alpha_2)$. As in the previous case we will want to use Lemma 3.2.4. Firstly then, h_3 is harmonic in $\mathcal{D}(0, \delta)$ since it is just the sum of v's taken at points avoiding the slits. Now,

$$(h_3)_\varphi(r)_{\text{out}} = (h_1)_\varphi(re^{i\psi_1})_{\text{out}} + (h_2)_\varphi(r)_{\text{out}} = (h_1)_\varphi(re^{i\psi_1}) + (h_2)_\varphi(r)$$

since h_1 and h_2 have two sided derivatives at $re^{i\psi_1}$ and r respectively. Looking back at the definitions of h_1 and h_2 we see that the derivative of h_2 vanishes at r and so

$$(h_3)_\varphi(r)_{\text{out}} = (h_1)_\varphi(re^{i\psi_1}) \leq 0$$

by (A.2). All we need now to apply the lemma is $(-h_3)_\varphi(re^{i\delta})_{\text{in}} \geq 0$. A calculation shows

$$(-h_3)_\varphi(re^{i\delta})_{\text{in}} = (-h_1)_\varphi(re^{i(\delta+\psi_1)})_{\text{in}} + (-h_2)_\varphi(re^{i\delta})_{\text{in}}.$$

Recalling the definition of ψ_1 it is easy to check that $\delta + \psi_1 = \frac{1}{2}(\alpha_4 - \alpha_3)$. This is the δ of the preceeding paragraph, so we know the h_1 term is non-negative. This leaves us to determine the sign of $(-h_2)_\varphi(re^{i\delta})_{\text{in}}$. By a similar calculation as above

$$h_2(re^{i(\delta-\varepsilon)}) - h_2(re^{i\delta}) = v(re^{i(\alpha_3-\varepsilon)}) + v(re^{i(\alpha_2+\varepsilon)}) - v(re^{i\alpha_3}) - v(re^{i\alpha_2}).$$

Once again, for $r \in K$ the terms $v(re^{i\alpha_3})$ and $v(re^{i\alpha_2})$ are zero, so $(-h_2)_\varphi(re^{i\delta})_{\text{in}} \geq 0$ and we are done. If $r \notin K$ then v has two-sided derivatives at $re^{i\alpha_3}$ and $re^{i\alpha_2}$. This reduces showing $(-h_2)_\varphi(re^{i\delta})_{\text{in}} \geq 0$ to showing

$$v_\varphi(re^{i\alpha_2}) - v_\varphi(re^{i\alpha_3}) \geq 0. \tag{A.3}$$

In Baerstein's paper [5], The α_3 is mistaken for an α_1 in (A.3) and proof of this mistaken inequality is given. The proof of this corrected statement involves reflecting the v in a line passing through e^{im_2} and considering the function

$$f(re^{i\theta}) = v(re^{i(m_2-\theta)}) + v(re^{i(m_2+\theta)})$$

for $\theta \in [-\delta, \delta + (\alpha_2 - \alpha_1)]$. It should shown that

$$f(re^{i(-\delta)}) = f(re^{i\delta}) \leq f(re^{i(\delta+\alpha_2-\alpha_1)})$$

for then, by Lemma 3.2.5 we would have

$$f_\theta(re^{i\delta}) = -v_\varphi(re^{i\alpha_2}) + v_\varphi(re^{i\alpha_3}) \geq 0$$

which, by reflecting back through the line containing e^{im_2} gives (A.3). Hence v_1 is superharmonic in $\mathcal{D}(\theta_1, \theta_2)$.

So far we have shown that v_1 is superharmonic in

$$\mathcal{D}(0, \theta_1) \cup \mathcal{D}(\theta_1, \theta_2) \cup \mathcal{D}(\theta_2, \frac{\pi}{3}).$$

To extend this superharmonicty to all of $\mathcal{D}(0, \frac{\pi}{3})$ we can use Lemma 3.2.3, provided we can show that

$$(v_1)_\theta(re^{i\theta_1})_{\text{in}} \geq (v_1)_\theta(re^{i\theta_1})_{\text{out}} \tag{A.4}$$

and

$$(v_1)_\theta(re^{i\theta_2})_{\text{in}} \geq (v_1)_\theta(re^{i\theta_2})_{\text{out}}. \tag{A.5}$$

Using the definition of the in derivative and v_1 we see that

$$\begin{aligned}
(v_1)_\theta(re^{i\theta})_{\text{in}} &= \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{\varepsilon}\left(v_1(re^{i\theta_1}) - v_1(re^{i(\theta_1-\varepsilon)})\right)\\
&= \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{\varepsilon}\left(\int_{E(\theta_1)} v(re^{i\varphi})\,d\varphi - \int_{E(\theta_1-\varepsilon)} v(re^{i\varphi})\,d\varphi\right)\\
&= \liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{\varepsilon}\left(\int_{m_3-3\theta_1}^{m_3+3\theta_1} v(re^{i\varphi})\,d\varphi - \int_{m_3-3(\theta_1+\varepsilon)}^{m_3+3(\theta_1-\varepsilon)} v(re^{i\varphi})\,d\varphi\right)\\
&= 3\liminf_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \frac{1}{3\varepsilon}\left(\int_{m_3-\psi_1}^{m_3-\psi_1+3\varepsilon} v(re^{i\varphi})\,d\varphi - \int_{m_3+\psi_1-3\varepsilon}^{m_3+\psi_1} v(re^{i\varphi})\,d\varphi\right)\\
&= 3v(re^{i(m_3-\psi_1)}) + 3v(re^{i(m_3+\psi_1)}) = 3h_1(re^{i\psi_1})
\end{aligned}$$

recalling that $\psi_1 = 3\theta_1$. Several more computations of similar style give us

$$\begin{aligned}
(v_1)_\theta(re^{i\theta_1})_{\text{in}} &= 3h_1(re^{i\psi_1}),\\
(v_1)_\theta(re^{i\theta_1})_{\text{out}} &= \frac{3}{2}(h_1(re^{i\psi_1}) + h_2(r)),\\
(v_1)_\theta(re^{i\theta_2})_{\text{in}} &= \frac{3}{2}h_3(re^{i\psi_3}),\\
(v_1)_\theta(re^{i\theta_2})_{\text{out}} &= h_3(re^{i\psi_3}) + 2u(re^{im_1}).
\end{aligned}$$

Using these we can rewrite the inequatlities in (A.4) and (A.5) as

$$h_2(r) \leq h_1(re^{i\psi}) \quad , \quad v(re^{im_3}) \leq \frac{1}{4}h_3(re^{i\psi_3}), \tag{A.6}$$

for all $r \in (0,1)$. We now give proofs of these inequalities.

Let $\delta = \frac{1}{2}(\alpha_3 - \alpha_2)$ and consider $h_2(re^{i\varphi})$ and $h_1(re^{i(\varphi+\psi_1)})$ on the sector $\mathcal{D}(-\delta,\delta)$. Both these functions are equal to 2 on the circular part of the boundary of this sector and

$$h_2(re^{i(-\delta)}) = h_2(re^{i\delta}) = v(re^{i\alpha_2}) + v(re^{i\alpha_3}).$$

We saw earlier in this proof that $(h_1)_\varphi(re^{i\varphi}) \leq 0$ for $0 < \varphi < \frac{1}{2}(\alpha_4 - \alpha_3)$. Also,

$$h_1(re^{i\theta}) = v(re^{i(m_3+\theta)}) + v(re^{i(m_3-\theta)}) = h_1(re^{i(-\theta)}),$$

so h_1 is symmetric decreasing about the positive real axis. Thus,

$$h_1(re^{i(-\delta+\psi_1)}) \geq h_1(re^{i(\delta+\psi_1)}) = v(re^{i\alpha_3}) + v(re^{i\alpha_4}).$$

By Lemma 3.2.6 we know $v(re^{i\alpha_4}) \geq v(re^{i\alpha_2})$, therefore $h_1(re^{i(\varphi+\psi_1)}) \geq h_2(re^{i\varphi})$ for $\varphi = \pm\delta$ and $r \in (0, 1)$. We know h_1 and h_2 are harmonic on $\mathcal{D}(-\delta, \delta)$. Applying the maximum principle to $h_2(re^{i\varphi}) - h_1(re^{i(\varphi+\psi_1)})$ tells us that this function is greater than zero inside $\mathcal{D}(-\delta, \delta)$, and therefore at $\varphi = 0$ giving the first inequality of (A.6).

The proof of the second inequality follows a similar pattern; let $\delta = \frac{1}{2}(\alpha_2 - \alpha_1)$ and compare $\frac{1}{2}h_3(re^{i(\varphi+\psi_3)})$ with $v(re^{i(m_1+\varphi)}) + v(re^{i(m_1-\varphi)})$ on $\mathcal{D}(-\delta, \delta)$.

Tracing back through this argument we see that v_1 is superharmonic on $\mathcal{D}(0, \frac{\pi}{3})$. $\square$

Appendix B

Maple code for numdp

```
# NUMDP.TXT
#   This is some Maple code to solve a numerical Dirichlet problem on a slit
#   annulus. The functions in this package are described below:
#
# numdp(M,N,Srow,Scol)
#   Sets up and solves a discrete Dirichlet problem:
#    numdp(M,N,Srow,Scol) := dpsolve(init_grid(M,N,Srow,Scol)
#
# init_grid(M,N,l1,l2)
#   Initializes the grid upon which the discrete Dirichlet problem is
#   described.
# INPUT:
#   M,N --- postive integers giving size of grid. (less than 50)
#   l1 --- list of integers 0 < j < M-1 defining Srow
#   l2 --- list of integers 0 <= k < N defining Scol
# OUTPUT:
#   P --- a structure containing the relevant information
#
# dpsolve(P)
#   Solves the Dirichlet problem described by P.
# INPUT:
#   P --- Grid data structure returned from init_grid
# OUTPUT:
#   U --- an M by N matrix containing the solution of the problem
#

numdp := proc(M,N,Srow,Scol)
    RETURN(dpsolve(init_grid(M,N,Srow,Scol)));
```

```
end;

init_grid := proc(M,N,Srow,Scol)
local U,j,k;
    if (1<=M) and (M<=50) then
        if (1<=N) and (N<=50) then
            U:=array(0..M-1,0..N-1);
        else
            ERROR(`Illegal value for grid width.`);
        fi;
    else
        ERROR(`Illegal value for grid height.`);
    fi;

    for k from 0 to N-1 do
        U[0,k] := 1;
    od;

    for j in Srow do
        if (0<j) and (j<M-1) then
            for k in Scol do
                if (0<=k) and (k<N) then
                    U[j,k] := 0;
                else
                    ERROR(`Illegal entry in Scol`);
                fi;
            od;
        else
            ERROR(`Illegal entry in Srow`);
        fi;
    od;

    RETURN([U,M,N,Srow,Scol]);
end;

dpsolve := proc(P)
local eqn,c,M,N,Srow,Scol,U,i,j,k,tj,bj,lk,rk,v,aug,A,b,x,numcols;
    U:=op(P)[1];
    M:=op(P)[2];
    N:=op(P)[3];
    Srow:=op(P)[4];
    Scol:=op(P)[5];
```

```
eqn:=[];
for j from 1 to M-2 do
    for k from 0 to N-1 do
        if not(member(j,Srow) and member(k,Scol)) then
            tj := j-1 mod M;
            bj := j+1 mod M;
            lk := k-1 mod N;
            rk := k+1 mod N;
            eqn:=[op(eqn), 4*U[j,k] - (U[tj,k]+U[bj,k]+U[j,lk]+U[j,rk]) = 0];
        fi;
    od;
od;

for k from 0 to N-1 do
    eqn := [op(eqn),U[M-1,k] - c = 0];
od;
print(`Solving problem on `,M,N,` grid.`);
print(`Number of equations: `,nops(eqn));

eqn := [op(eqn),sum(U[M-2,l],'l'=0..N-1) - N*c = 0];

v:=[];
for j from 0 to M-1 do
    for k from 0 to N-1 do
        if not(assigned(U[j,k])) then
            v:=[op(v),U[j,k]];
        fi;
    od;
od;
v:=[op(v),c];

aug := linalg[genmatrix](eqn,v,0);
numcols := linalg[coldim](aug);
b := linalg[col](aug,numcols);
A := linalg[delcols](aug,numcols..numcols);
x := linalg[linsolve](A,b);

for i from 1 to nops(v) do
    assign(v[i] = x[i]);
od;
U:=convert(U,matrix);
RETURN(U);
```

```
end;

output3d := proc(U,filename,slitval)
local j,k,jmm,r,theta,M,N,x,y,z;
    readlib(write):
    open(filename);
    M:=linalg[coldim](U);
    N:=linalg[rowdim](U);
    for k from 0 to N-1 do
        r := exp(-2*k*Pi/N);
        for j from 0 to M do
            theta := 2*Pi*j/M;
            x := evalf(r*cos(theta));
            y := evalf(r*sin(theta));
            jmm := j mod M;
            if slitval=0 then
                z := evalf(U[k+1,jmm+1]);
            else
                z := evalf(1-U[k+1,jmm+1]);
            fi;
            writeln(x,y,z);
        od;
        writeln();
    od;
    close(filename);
end;

outputgrid := proc(U, filename)
local j,k,M,N,approx;
   readlib(write):
   M := linalg[coldim](U);
   N := linalg[rowdim](U);
   open(filename);
   for j from 1 to N do
       for k from 1 to M do
           approx := evalf(U[j,k]);
           write(approx);
       od;
       writeln();
   od;
   close(filename);
end;
```

```
star := proc(U)
local M,N,j,k,sortv,starv,V;
   V := [];
   M := linalg[rowdim](U);
   N := linalg[coldim](U);
   for j from 1 to M do
      sortv := map(neg,sort(map(neg,convert(linalg[row](U,j),list))));
      for k from 1 to N do
         starv[k] := sum(sortv['i'],'i'=1..k)
      od;
      V := [op(V),convert(starv,list)];
   od;
   RETURN(convert(V,matrix));
end;

neg := proc(x)
   RETURN(-x)
end;
```

Appendix C

Fortran code for PLTMG

C.1 Problem Definition

```
c************************** file: slit4.f ***************************
c--------------------------------------------------------------------
c
c          piecewise linear triangle multi grid package
c
c                  edition 7.1 - - - june, 1994
c
c                  problem name  - - - slit4
c
c                    Author - - - Mark Reid
c                  Created  - - -  20th August 1996
c
c--------------------------------------------------------------------
c
        double precision function a1xy(x,y,u,ux,uy,rl,itag,itype)
c
            implicit double precision (a-h,o-z)
            implicit integer (i-n)
c
        go to (1,2,3,2,2),itype
    1   a1xy=ux
        return
    2   a1xy=0.0d0
        return
    3   a1xy=1.0d0
```

```
      return
      end
c
      double precision function a2xy(x,y,u,ux,uy,rl,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
c
      go to (1,2,2,3,2),itype
    1 a2xy=uy
      return
    2 a2xy=0.0d0
      return
    3 a2xy=1.0d0
      return
      end
c
      double precision function fxy(x,y,u,ux,uy,rl,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
c
      fxy=0.0d0
      return
      end
c
      double precision function gxy(x,y,u,rl,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
          common /atest2/iu(100),ru(100)
c
      go to (2,2,2,1,2,2),itype
 1    if(itag.eq.1) then
          gxy=1.0d0
      else
          gxy=0.0d0
      endif
      return
 2    gxy=0.0d0
      return
      end
c
```

```
      double precision function p1xy(x,y,u,ux,uy,rl,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
c
    5 p1xy=0.0d0
      return
      end
c
      double precision function uxy(x,y,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
          common /atest2/iu(100),ru(100)
c
      uxy=0.0d0
      return
      end
c
      double precision function p2xy(x,y,dx,dy,u,ux,uy,rl,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
c
      p2xy=0.0d0
      return
      end
c
      double precision function qxy(x,y,dx,dy,u,ux,uy,rl,itag,itype)
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
c
      qxy=0.0d0
      return
      end
c
      subroutine usrcmd
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)

          common /atest1/ip(100),rp(100)
```

```
           common /atest2/iu(100),ru(100)
c
c       call reset(table,alias,indx,ni,nr,iu,ru)
        return
        end
c
c
        subroutine gdata(title,vx,vy,xm,ym,itnode,
     +          ibndry,jb,ip,rp,iu,ru,w)
c
            implicit double precision (a-h,o-z)
            implicit integer (i-n)
            integer
     +          itnode(5,*),ibndry(6,*),jb(*),ip(100),iu(100),
     +           prblm(10)
            double precision
     +          vx(*),vy(*),xm(*),ym(*),rp(100),ru(100),w(*),
     +          thpi(10),lenth(10)
            character*80
     +          title
            save ispd,iprob,thpi,lenth
c
            data ispd,iprob/0,8/
c
        title='Triangulation Scheme III'
c
c Initialize variables and arrays
        nsli=iu(1)
        nti=4*nsli
        nvi=4*nsli+1
        nbi=4*nsli
        nci=1
c
        do i=1,nsli
           thpi(i)=ru(i)
           lenth(i)=ru(i+nsli)
        enddo
        thpi(nsli+1)=ru(1)+2.0d0
c
c  Here are some variable that specify the position of the different
c  points within the vx, vy arrays.
c  na1p + i -- upper point corresponding to thpi(i)
c  na2p + i -- lower point corresponding to thpi(i)
```

```
c  nmp + i -- the midpoint on the circle between thpi(i) an thpi(i+1)
c  npp + i -- the end of the slit starting at thpi(i)
      na1p=1
      na2p=nsli+1
      nmp=2*nsli+1
      npp=3*nsli+1
c  Set the correct variables in the IP array.
      ip(1)=nti
      ip(2)=nvi
      ip(3)=nci
      ip(4)=nbi
      ip(5)=1
      ip(6)=iprob
      ip(8)=ispd
      ip(14)=nvi
      ip(74)=1
c
      pi=3.141592653589793d0
      do i=1,nsli
c          ** Set first triangle in each region **
          itnode(1,4*i-3)=na1p+i
          itnode(2,4*i-3)=nmp+i
          itnode(3,4*i-3)=npp+i
          itnode(4,4*i-3)=1
          itnode(5,4*i-3)=0
c          ** Set first boundary in each region **
          ibndry(1,4*i-3)=npp+i
          ibndry(2,4*i-3)=na1p+i
          ibndry(3,4*i-3)=0
          ibndry(4,4*i-3)=-1
          ibndry(5,4*i-3)=1
          ibndry(6,4*i-3)=0
c          ** Set second triangle in each region **
          itnode(1,4*i-2)=nmp+i
          itnode(2,4*i-2)=npp+i
          itnode(3,4*i-2)=1
          itnode(4,4*i-2)=2
          itnode(5,4*i-2)=0
c          ** Set second boundary in each region **
          ibndry(1,4*i-2)=na1p+i
          ibndry(2,4*i-2)=nmp+i
          ibndry(3,4*i-2)=1
          ibndry(4,4*i-2)=-1
```

```
          ibndry(5,4*i-2)=0
          ibndry(6,4*i-2)=0
c         ** Set third triangle in each region **
          itnode(1,4*i-1)=1
          itnode(2,4*i-1)=nmp+i
          if(i.eq.nsli) then
             itnode(3,4*i-1)=npp+1
          else
             itnode(3,4*i-1)=npp+i+1
          endif
          itnode(4,4*i-1)=3
          itnode(5,4*i-1)=0
c         ** Set third boundary in each region **
          ibndry(1,4*i-1)=nmp+i
          ibndry(2,4*i-1)=na2p+i
          ibndry(3,4*i-1)=1
          ibndry(4,4*i-1)=-1
          ibndry(5,4*i-1)=0
          ibndry(6,4*i-1)=0
c         ** Set fourth triangle in each region **
          if(i.eq.nsli) then
              itnode(1,4*i)=npp+1
          else
              itnode(1,4*i)=npp+i+1
          endif
          itnode(2,4*i)=nmp+i
          itnode(3,4*i)=na2p+i
          itnode(4,4*i)=4
          itnode(5,4*i)=0
c         ** Set fourth boundary in each region **
          ibndry(1,4*i)=na2p+i
          if(i.eq.nsli) then
              ibndry(2,4*i)=npp+1
          else
              ibndry(2,4*i)=npp+i+1
          endif
          ibndry(3,4*i)=0
          ibndry(4,4*i)=-1
          ibndry(5,4*i)=1
          ibndry(6,4*i)=0
      enddo
c
      vx(1)=0.0d0
```

```
      vy(1)=0.0d0
      do i=1,nsli
          arg1=pi*thpi(i)
          arg2=pi*thpi(i+1)
          arg3=(arg1+arg2)/2.0d0
          vx(na1p+i)=dcos(arg1)
          vy(na1p+i)=dsin(arg1)
          vx(na2p+i)=dcos(arg2)
          vy(na2p+i)=dsin(arg2)
          vx(nmp+i)=dcos(arg3)
          vy(nmp+i)=dsin(arg3)
          vx(npp+i)=(1.0d0-lenth(i))*dcos(arg1)
          vy(npp+i)=(1.0d0-lenth(i))*dsin(arg1)
      enddo
      xm(1)=0.0d0
      ym(1)=0.0d0
      return
      end
c
```

C.2 Maximal Arrangement Search

```
c************************* file: varlength.f ************************
c-----------------------------------------------------------------------
c
c           piecewise linear triangle multi grid package
c
c                   edition 7.1 - - - september, 1996
c
c-----------------------------------------------------------------------
      program mytest
c
c     storage allocation
c
          implicit double precision (a-h,o-z)
          implicit integer (i-n)
c
          parameter (lenw=300000,maxv=6000,maxt=16000,maxc=500,
     +        maxb=5000,maxjb=10000,nsli=3,ncthstep=100,nclenstep=100)

          integer
```

```
     +          itnode(5,maxt),ibndry(6,maxb),jb(maxjb),filnum
          double precision
     +         w(lenw),vx(maxv),vy(maxv),xm(maxc),ym(maxc),dummy(100),
     +           thstep(nsli),initth(nsli),finth(nsli),
     +           x(1),y(1),u(1),
     +           maxth1(nclenstep),maxth2(nclenstep),maxval(nclenstep),
     +           lenth,initlenth,finlenth,lenstep
          character*80
     +          title,cdummy
          character*1
     +          prompt
c
          common /atest1/ip(100),rp(100)
          common /atest2/iu(100),ru(100)
          common /atest3/icrtr,icrtw,ifilrw,jnlrw,jnlst
c
          external a1xy,a2xy,fxy,gxy,uxy,p1xy,p2xy,qxy
c
c       default settings for parameters
c
          data ispd,iprob,iadapt,nvtrgt/0,0,1,1/
          data ifirst,level1,irefn,idbc/1,1,5,0/
          data itmax,maxm/20,20/
          data rtrgt,rltrgt/0.0d0,0.0d0/
c
          data inplsw,igrsw,mag,ix,iy/0,14,1,1,1/
          data ifun,ncon,iscale,lines,numbrs/0,11,0,0,0/
          data nx,ny,nz/0,0,1/
          data smin,smax/0.0d0,0.0d0/
c
          data kscale,kfun,nrgn/0,0,10/
          data fract,tol/0.0d0,0.02d0/
          data grade,hmax/1.5d0,0.0d0/
c
          data iunit,jnlnum/2,-99/
c
          data filnum,prompt/12,'+'/
          data mxcolr,idevce/232,1/
c
          data initth/0.0d0, 0.1d0, 0.1d0/
          data finth/0.0d0, 1.9d0, 1.9d0/
          data initlenth,finlenth/0.1d0, 0.9d0/
          data ratio/1.0d0/
```

```
c
c       device numbers for files etc.
c
        icrtr=5
        icrtw=6
        ifilrw=14
        jnlrw=13
        jnlst=0
c
c       initialize the ip, rp, iu, and ru arrays
c
        do i=1,100
            ip(i)=0
            iu(i)=0
            rp(i)=0.0d0
            ru(i)=0.0d0
        enddo
c
        title='pltmg'
c
c       parameters for pltmg
c
        ip(5)=ifirst
        ip(6)=iprob
        ip(7)=idbc
        ip(8)=ispd
        ip(9)=level1
        ip(10)=maxm
        ip(11)=itmax
        ip(12)=irefn
        ip(13)=iadapt
        ip(14)=nvtrgt
        ip(15)=iunit
c
        rp(1)=rltrgt
        rp(2)=rtrgt
c
c       storage parameters
c
        ip(20)=lenw
        ip(21)=maxt
        ip(22)=maxv
        ip(23)=maxc
```

```
      ip(24)=maxb
      ip(25)=maxjb
c
c     parameters for atest
c
      ip(26)=jnlnum
      ip(27)=filnum
      ip(28)=1
      if(jnlnum.ne.jnlst) call journl(ip)
c
c     parameters for trigen and skeltn
c
      ip(55)=kfun
      ip(56)=nrgn
      ip(57)=kscale
c
      rp(3)=hmax
      rp(4)=grade
      rp(5)=fract
      rp(6)=tol
c
c     parameters for triplt, gphplt, and inplt
c
      ip(60)=inplsw
      ip(61)=igrsw
      ip(62)=mag
      ip(63)=ix
      ip(64)=iy
      ip(65)=ifun
      ip(66)=ncon
      ip(67)=iscale
      ip(68)=lines
      ip(69)=numbrs
      ip(70)=nx
      ip(71)=ny
      ip(72)=nz
      ip(73)=mxcolr
      ip(74)=idevce
c
      rp(7)=smin
      rp(8)=smax

      open(unit=2,file='crap.out',status='NEW')
```

```
c ********************************************************
c RATIO SCAN FOR MAXIMAL VALUES
c     This code will do a huge scan of a large sample of
c     possible positions and ratios for a simply connected
c     three slit (2 of equal length) domain.
c
c     Set defaults
        nthstep = ncthstep
        nlenstep = nclenstep
c  Read in arguments from standard input
        read*, initth(2),finth(2)
        read*, initth(3),finth(3),nthstep
        read*, initlenth,finlenth,nlenstep
        read*, ratio

c  Initialize IU arrays
        iu(1)=nsli
        pi=3.141592653589793d0
c
c  Calculate increment steps
        do i=1,nsli
           thstep(i)=(finth(i)-initth(i))/dfloat(nthstep-1)
        enddo
        lenstep=(finlenth - initlenth)/(dfloat(nlenstep-1))
c
        print*,'# Ratio ',ratio
        print*,'# Initial (fixed) Length ',initlenth
        print*,'# Final (fixed) Length ',finlenth
        print*,'# Length step ',nlenstep
        print*,'# Initial Theta 1 ',initth(2)
        print*,'# Final Theta 1 ',finth(2)
        print*,'# Initial Theta 2 ',initth(3)
        print*,'# Final Theta 2 ',finth(3)
        print*,'# Theta step ',nthstep
c
c  Loop paramters around -- results in Unit 1
        lenth=initlenth
        lencnt=1
 10     if (lenth.le.finlenth) then
           ru(nsli+1)=lenth
           ru(nsli+2)=lenth*ratio
           ru(nsli+3)=lenth*ratio
           maxth1(lencnt)=0.0d0
```

```
         maxth2(lencnt)=0.0d0
         maxval(lencnt)=0.0d0
c
         pslit1=initth(1)
         pslit2=initth(2)
         num2=1
c
 20      if (pslit2.le.finth(2)) then
            pslit3=initth(3)
            num3=1
c
 30         if (pslit3.le.pslit2) then
               pslit3=initth(3)+dfloat(num3)*thstep(3)
               num3 = num3 +1
               goto 30
            endif
 40         if (pslit3.le.finth(3)) then
c
               ru(1)=pslit1
               ru(2)=pslit2
               ru(3)=pslit3
c
               ip(28)=1
               call gdata(title,vx,vy,xm,ym,itnode,ibndry,
     +              jb,ip,rp,iu,ru,w)
               ip(28)=0
               call setw(vx,vy,xm,ym,itnode,ibndry,jb,
     +              ip,w)
               ip(5)=ifirst
               call pltmg(vx,vy,xm,ym,itnode,ibndry,ip,rp,
     +              w,a1xy,a2xy,fxy,gxy,p1xy,p2xy)
               ip(17)=1
               x(1)=0.0d0
               y(1)=0.0d0
               u(1)=1.0d0
               ip(16)=1
               call pltevl(x,y,u,ux,uy,vx,vy,xm,ym,
     +              itnode,ibndry,ip,rp,w)
c
               if(u(1).gt.maxval(lencnt)) then
                  maxval(lencnt)=u(1)
                  maxth1(lencnt)=ru(2)
                  maxth2(lencnt)=ru(3)
```

```
                endif
                pslit3=initth(3)+dfloat(num3)*thstep(3)
                num3=num3+1
                goto 40
             endif
             pslit2 = initth(2) + dfloat(num2)*thstep(2)
             num2 = num2 + 1
             goto 20
          endif
          print*,ru(nsli+1),maxth1(lencnt),maxth2(lencnt),
     +         maxval(lencnt)
          lenth = initlenth + dfloat(lencnt)*lenstep
          lencnt = lencnt + 1
          goto 10
       endif
       close(unit=2)
       stop
c
  301  format(/ '                   iflag =',i5,'        cflag =',i5)
  302  format(/ ' iflag =',i4,' cflag =',i4,'   nti =',i4,
     +     '   nvi =',i4,'    nci =',i4,'    nbi =',i4)
  303  format(/ ' iflag =',i4,' cflag =',i4,'   ntr =',i4,
     +     '   nvr =',i4,'    ncr =',i4,'    nbr =',i4)
  304  format(/ ' iflag =',i4,' cflag =',i4,'   ntf =',i4,
     +     '   nvf =',i4,'    nci =',i4,'    nbf =',i4)
       end
```

Bibliography

[1] L. Ahlfors, *Conformal invariants: Topics in geometric function theory*, McGraw-Hill, Inc., 1973.

[2] ______, *Complex analysis*, third ed., McGraw-Hill, Inc., 1979.

[3] A. Baernstein II, *Univalent functions and circular symmetrization*, Acta mathematica **18** (1975), 139–169.

[4] ______, *Dubinin's symmetrization theorem*, Springer Verlag Lecture Notes in Mathematics **1275** (1987), 23–30.

[5] ______, *On the harmonic measure of slit domains*, Complex Variables **9** (1987), 131–142.

[6] R. E. Bank, *Pltmg: A software package for solving elliptical partial differential equations - users' guide 7.0*, Society for Industrial and Applied Mathematics, 1994.

[7] R. Courant, *Dirichlet's principle*, Interscience Publishers, Inc., 1950.

[8] V. N. Dubinin, *Change of harmonic measure in symmetrization (russian)*, Mat. Sb. (N.S.) **2** (1984), 272–279.

[9] P. L. Duren, *Univalent functions*, Springer-Verlag New York Inc., 1983.

[10] Hayman, *Multivalent functions*, second ed., Cambridge University Press, 1994.

[11] Churchill R. & Brown J., *Complex variables and applications*, fifth ed., McGraw-Hill, Inc., 1990.

[12] P. Jones, *Lecture notes in potential theory*, Notes taken by C. Bishop.

[13] S. Kakutani, *Two-dimensional brownian motion and harmonic functions*, Proc. Imp. Acad. Tokyo **20** (1944), 706–14.

[14] Z. Nehari, *Conformal mapping*, Dover Publications, Inc., 1975.

[15] J. R. Quine, *Symmetrization inequalities for discrete harmonic functions*, Preprint.

[16] ———, *On the harmonic measure of slit domains*, Complex Variables **11** (1989), 223–231.

[17] W. Rudin, *Real and complex analysis*, second ed., McGraw-Hill, Inc., 1974.

[18] W. A. Strauss, *Partial differentail equations: An introduction*, John Wiley and Sons, Inc., 1992.