COMP2610/6261 — Information Theory

Lecture 13: Symbol Codes for Lossless Compression

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1. Variable-Length Codes
   - Unique Decodeability
   - Prefix Codes
2. The Kraft Inequality
3. Summary

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Codes: A Review

Notation:

- If $\mathcal{A}$ is a finite set then $\mathcal{A}^N$ is the set of all *strings of length* N.
- $\mathcal{A}^+ = \bigcup_N \mathcal{A}^N$ is the set of *all finite strings*

Examples:

- $\{0,1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$
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Binary Symbol Code

Let X be an ensemble with $\mathcal{A}_X = \{a_1, \ldots, a_I\}$.
A function $c : \mathcal{A}_X \to \{0,1\}^+$ is a **code** for X.

- The binary string $c(x)$ is the **codeword** for $x \in \mathcal{A}_X$

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  Shorthand: $\ell_i = \ell(a_i)$ for $i = 1 \ldots, I$.
- The **extension** of c assigns codewords to any sequence $x_1 x_2 \ldots x_N$ from $\mathcal{A}^+$ by $c(x_1 \ldots x_N) = c(x_1) \ldots c(x_N)$

$$X \text{ is an ensemble with } \mathcal{A}_X = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$$

Example 1 (Uniform Code):

- Let $c(\mathtt{a}) = 0001$, $c(\mathtt{b}) = 0010$, $c(\mathtt{c}) = 0100$, $c(\mathtt{d}) = 1000$

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Unique Decodeability

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A code c for X is **uniquely decodeable** if no two strings from $\mathcal{A}_X^+$ have the same codeword. That is, for all $\mathbf{x}, \mathbf{y} \in \mathcal{A}_X^+$

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$$c(\mathtt{aaa}) = c(\mathtt{d}) = 111 \quad \text{and} \quad c(\mathtt{ab}) = c(\mathtt{c}) = 110$$

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Why is unique decodeability useful for compression?

Prefix Codes

a.k.a *prefix-free* or *instantaneous* codes

There is an simple property of codes that *guarantees* unique decodeability.

Prefix

A codeword $\mathbf{c} \in \{0,1\}^+$ is said to be a **prefix** of another codeword $\mathbf{c}' \in \{0,1\}^+$ if there exists a string $\mathbf{t} \in \{0,1\}^+$ such that $\mathbf{c}' = \mathbf{ct}$.

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A code $C = \{c_1, \ldots, c_I\}$ is a **prefix code** if for every codeword $c_i \in C$ there is no prefix of c_i in C.

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- $C_2 = \{0, 10, 110, 111\}$ is prefix-free
- $C_2' = \{1, 10, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 c_2$

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- $C_1 = \{0001, 0010, 0100, 1000\}$ is prefix-free
- $C_2 = \{0, 10, 110, 111\}$ is prefix-free
- $C_2' = \{1, 10, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 c_2$
- $C_2'' = \{1, 01, 110, 111\}$ is *not* prefix free since $c_3 = 110 = c_1 10$

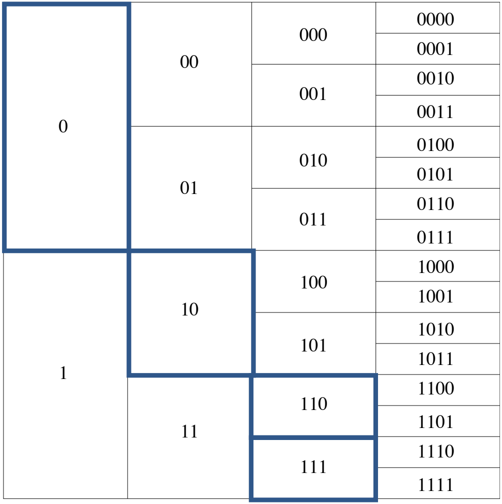
Prefix Codes as Trees

$$C_1 = \{0001, 0010, 0100, 1000\}$$

0	00	000	0000
			0001
		001	**0010**
			0011
	01	010	**0100**
			0101
		011	0110
			0111
1	10	100	**1000**
			1001
		101	1010
			1011
	11	110	1100
			1101
		111	1110
			1111

Prefix Codes as Trees

$$C_2 = \{0, 10, 110, 111\}$$



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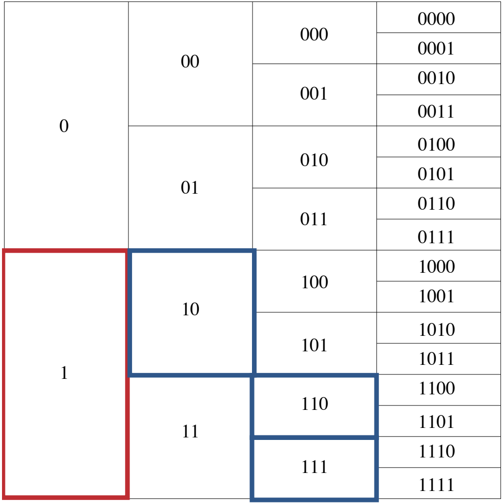
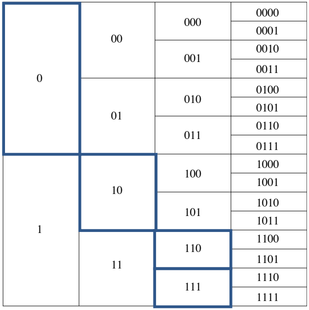


Prefix Codes are Uniquely Decodeable



- If $\ell^* = \max\{\ell_1, \ldots, \ell_I\}$ then symbol is decodeable after seeing at most ℓ^* bits
- Consider $C_2 = \{0, 10, 110, 111\}$
  - If $c(\mathbf{x}) = 0 \ldots$ then $x_1 = \texttt{a}$
  - If $c(\mathbf{x}) = 1 \ldots$ then $x_1 \in \{\texttt{b}, \texttt{c}, \texttt{d}\}$
  - If $c(\mathbf{x}) = 10 \ldots$ then $x_1 = \texttt{b}$
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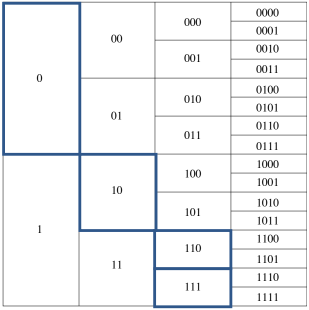

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However, not all uniquely decodeable codes are prefix codes

$C_3 = \{0, 01, 011, 111\}$ — Not prefix-free but uniquely decodeable Why?

Hint: Notice $c_3(\mathtt{bdca}) = 011110110$ and $c_2(\mathtt{acdb}) = 011011110$

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- Unique Decodeability
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2 The Kraft Inequality

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Lengths for Prefix Codes

Suppose someone said "I want codes with codewords lengths":

- $L_1 = \{4, 4, 4, 4\}$
- $L_2 = \{1, 2, 3, 3\}$
- $L_3 = \{2, 2, 3, 4, 4\}$
- $L_4 = \{1, 3, 3, 3, 3, 4\}$

Could you construct such codes? Uniquely Decodeable? Prefix-free?

0	00	000	0000
			0001
		001	0010
			0011
	01	010	0100
			0101
		011	0110
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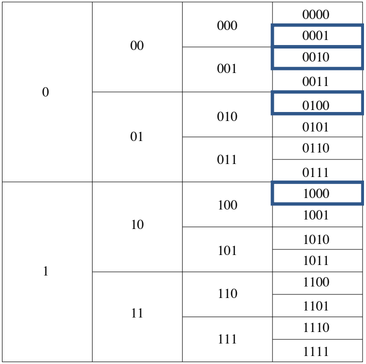

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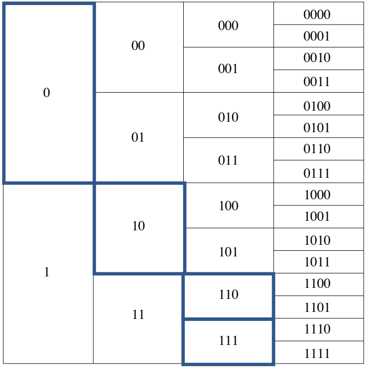

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<table>
<tr><td rowspan="8">0</td><td rowspan="4">00</td><td rowspan="2">000</td><td>0000</td></tr>
<tr><td>0001</td></tr>
<tr><td rowspan="2">001</td><td>0010</td></tr>
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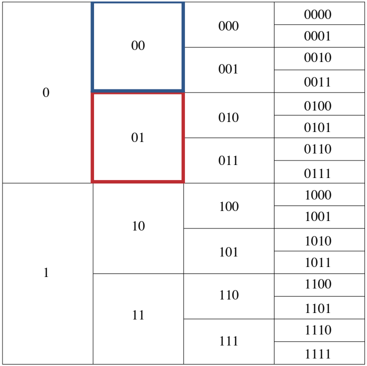

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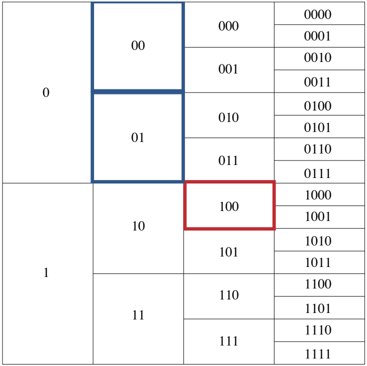

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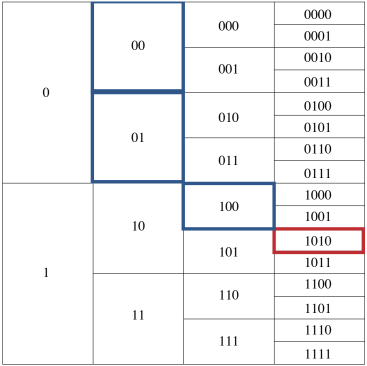

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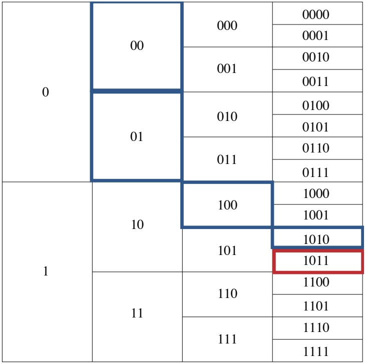
- $L_4 = \{1, 3, 3, 3, 3, 4\}$

Could you construct such codes? Uniquely Decodeable? Prefix-free?



Lengths for Prefix Codes

Suppose someone said "I want codes with codewords lengths":

- $L_1 = \{4, 4, 4, 4\}$ — $C_1 = \{0001, 0010, 0100, 1000\}$
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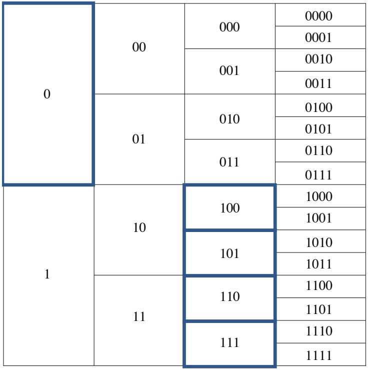
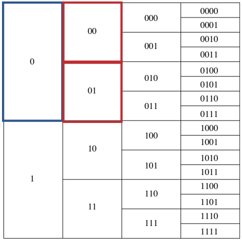


Prefixes Exclude Codes

Choosing a prefix codeword of length 1 — e.g., $c(\mathrm{a}) = 0$ — excludes:



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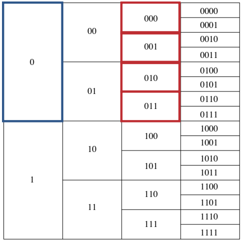

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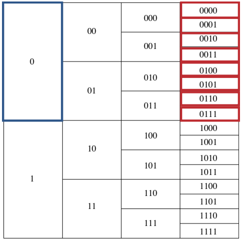

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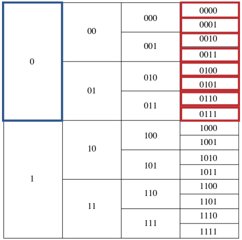

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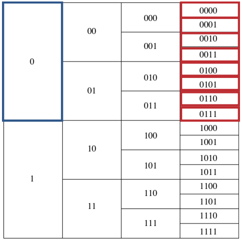

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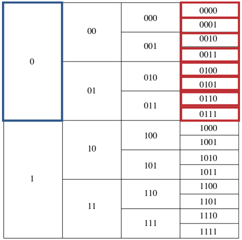

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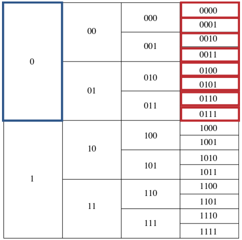

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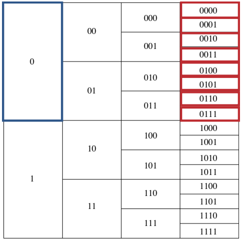

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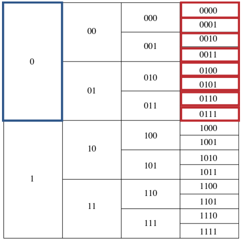

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The Kraft Inequality

a.k.a. The Kraft-McMillan Inequality

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For any prefix (binary) code C, its codeword lengths $\{\ell_1, \ldots, \ell_I\}$ satisfy

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Conversely, if the set $\{\ell_1, \ldots, \ell_I\}$ satisfy (1) then there exists a prefix code C with those codeword lengths.

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3. Lengths $\{1, 2, 2, 3\}$ give $\sum_{i=1}^{4} 2^{-\ell_i} = \frac{1}{2} + \frac{2}{4} + \frac{1}{8} > 1$ so no prefix code

Summary

Key ideas from this lecture:

- **Prefix** and **Uniquely Decodeable** variable-length codes
- Prefix codes are tree-like
- Every Prefix code is Uniquely Decodeable but not *vice versa*
- The **Kraft Inequality**:
  - Code lengths satisfying $\sum_i 2^{-\ell_i} \leq 1$ implies Prefix/U.D. code exists
  - Prefix/U.D. code implies $\sum_i 2^{-\ell_i} \leq 1$

Relevant Reading Material:

- MacKay: §5.1 and §5.2
- Cover & Thomas: §5.1, §5.2, and §5.5