Information Theory

Lecture 1: Introduction & Overview

Mark Reid

Research School of Computer Science
The Australian National University

Australian National University

28th November, 2014

Outline

1. Course Overview
   - What is Information?
   - Motivating Examples

2. Basic Concepts
   - Probability
   - Information and Entropy
   - Joint Entropy, Conditional Entropy and Chain Rule
   - Mutual Information, Divergence

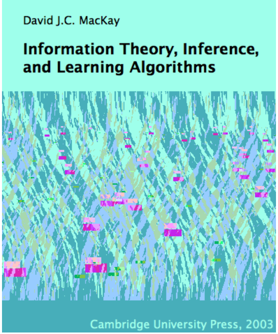
Overview

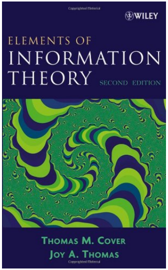
This short course is based on my COMP2610/COMP6261 course at ANU — a 26 hour, 2nd year undergraduate/Masters level course co-developed with **Aditya Menon** (NICTA) & **Edwin Bonilla** (NICTA).

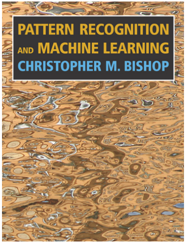
The ANU version of the course studies the fundamental limits and potential of the *representation* and *transmission* of information.

- Mathematical Foundations
- Coding and Compression
- Communication
- Probabilistic Inference
- Kolmogorov Complexity

TextBook



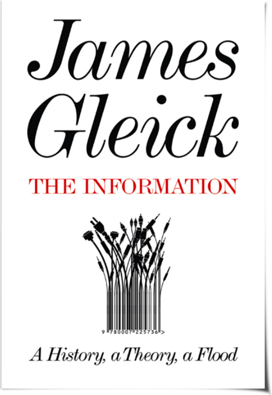




Mackay (ITILA, 2006) available online:
`http://www.inference.phy.cam.ac.uk/mackay/itila`

David MacKay's Lectures:
`http://www.inference.phy.cam.ac.uk/itprnn_lectures/`

The History of Information Theory



&

Information Theory and the Digital Age

by Aftab, Cheung, Kim, Thakkar, and Yeddanapudi.

`http://web.mit.edu/6.933/www/Fall2001/Shannon2.pdf`

Uses of Information Theory

- Statistical physics (thermodynamics, quantum information theory);
- Computer science (machine learning, algorithmic complexity, resolvability);
- Probability theory (large deviations, limit theorems);
- Statistics (hypothesis testing, multi-user detection, Fisher information, estimation);
- Economics (gambling theory, investment theory);
- Biology (biological information theory);
- Cryptography (data security, watermarking);
- Networks (self-similarity, traffic regulation theory).

What Is Information? (1)

According to a dictionary definition, information can mean

1. Facts provided or learned about something or someone:
   a vital piece of information.
2. What is conveyed or represented by a particular arrangement or sequence of things:
   genetically transmitted information.

In this course: information in the context of *communication*:

- Explicitly include uncertainty, modelled probabilistically
- Shannon (1948): "Amount of unexpected data a message contains"
  - A theory of information transmission
  - Source, destination, transmitter, receiver

What is Information? (2)

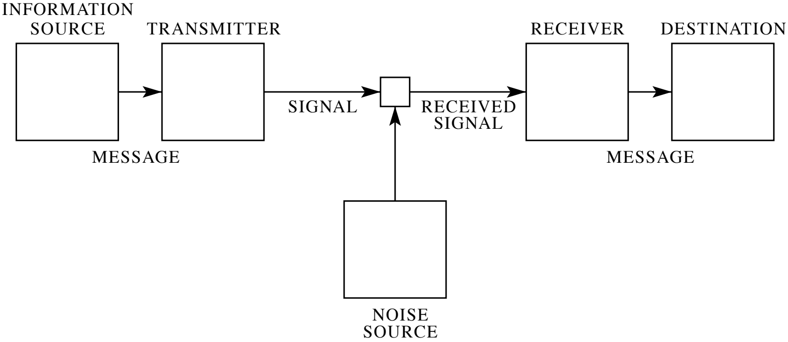


Fig. 1 — Schematic diagram of a general communication system.

From Shannon (1948)

What Is Information? (3)

Information is a message that is *uncertain* to receivers:

- If we receive something that we already knew with absolute certainty then it is non-informative.
- Uncertainty is crucial in measuring information content
- We will deal with uncertainty using probability theory

Information Theory

Information theory is the study of the fundamental *limits* and *potential* of the *representation* and transmission of information.

Examples

Example 1: What Number Am I Thinking of?

- I have in mind a number that is between 1 and 20
- You are allowed to ask me one question at a time
- I can only answer yes/no
- Your goal is to figure out the number as quickly as possible
- What strategy would you follow?

Example 1: What Number Am I Thinking of?

- I have in mind a number that is between 1 and 20
- You are allowed to ask me one question at a time
- I can only answer yes/no
- Your goal is to figure out the number as quickly as possible
- What strategy would you follow?

Your strategy + my answers = a code for each number

Some variants:

- What if you knew I was twice as likely to pick numbers more than 10?
- What if you knew I never chose prime numbers?
- What if you knew I only ever chose one of 7 or 13?

What is the optimal strategy/coding?

Example 2: Redundancy and Compression

Cn y rd ths sntnc wtht ny vwls?

Example 2: Redundancy and Compression

Cn y rd ths sntnc wtht ny vwls?

Can you read this sentence without any vowels?

Written English (and other languages) has much *redundancy*:

- Approximately 1 bit of information per letter
- Naively there should be almost 5 bits per letter

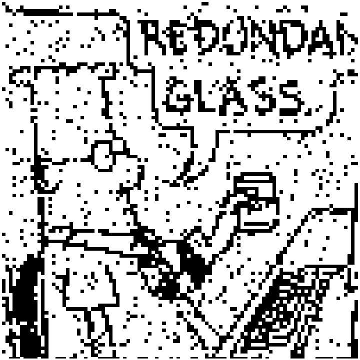
(For the moment think of “bit” as “number of yes/no questions”)

How much redundancy can we *safely* remove?

(Note: “rd” could be “read”, “red”, “road”, etc.)

Example 3: Error Correction

Hmauns hvae the aitliby to cerroct for eorrrs in txet and iegmas.



How much noise is it possible to correct for and how?

Overview of ANU Course

- How can we quantify information?
  - Probability, Basic Properties
  - Entropy & Information, Results & Inequalities
- How can we make good guesses?
  - Probabilistic Inference
  - Bayes Theorem and Applications
- How much redundancy can we safely remove?
  - Compression
  - Source Coding Theorems, Kraft Inequality
  - Block, Huffman, and Lempev-Ziv Coding
- How much noise can we correct and how?
  - Noisy-Channel Coding
  - Repetition Codes, Hamming Codes
- What is randomness?
  - Kolmogorov Complexity & Algorithmic Information Theory

Overview of ANU Course

- How can we quantify information?
  - Probability, Basic Properties
  - Entropy & Information, Results & Inequalities
- How can we make good guesses?
  - Probabilistic Inference
  - Bayes Theorem and Applications
- How much redundancy can we safely remove?
  - Compression
  - Source Coding Theorems, Kraft Inequality
  - Block, Huffman, and Lempev-Ziv Coding
- How much noise can we correct and how?
  - Noisy-Channel Coding
  - Repetition Codes, Hamming Codes
- What is randomness?
  - Kolmogorov Complexity & Algorithmic Information Theory
- Applications to Machine Learning
  - Max. entropy, online learning, & more

Overview of Short Course

- **Day 1**: Overview & Basic Concepts
  - Definitions: Probability, Entropy, Information, Divergence
  - Basic Properties & Relationships
- **Day 2**: Inequalities & Key Results
  - Probabilistic Inequalities
  - Information Theoretic Inequalities
  - Source Coding Theorems
  - Noisy-Channel Coding Theorem
- **Day 3**: Information Theory & Machine Learning
  - Online Learning
  - Exponential Families
  - Clustering

Outline

1. Course Overview

2. Basic Concepts
   - Probability
   - Information and Entropy
   - Joint Entropy, Conditional Entropy and Chain Rule
   - Mutual Information, Divergence

Probability

Let X, Y be random variables taking values in $\{x_i\}_{i=1}^N$ and $\{y_j\}_{j=1}^M$ (resp.)

Sum Rule / Marginalization :

$$\overbrace{p(X = x_i)}^{\textit{marginal}} = \sum_j \overbrace{p(X = x_i, Y = y_j)}^{\textit{joint}}$$

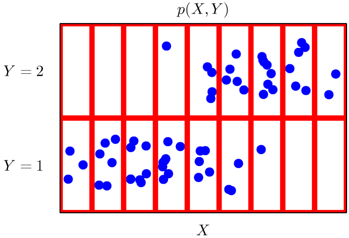
Product Rule :

$$\begin{aligned}\overbrace{p(X = x_i, Y = y_j)}^{\textit{joint}} &= \overbrace{p(Y = y_j | X = x_i)}^{\textit{conditional}} \overbrace{p(X = x_i)}^{\textit{marginal}} \\ &= p(X = x_i | Y = y_j) p(Y = y_j)\end{aligned}$$

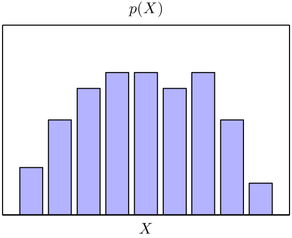
Bayes Rule :

$$\overbrace{p(Y = y | X = x)}^{\textit{posterior}} = \frac{\overbrace{p(X = x | Y = y)}^{\textit{likelihood}} \overbrace{p(Y = y)}^{\textit{prior}}}{\underbrace{p(X = x)}_{\textit{evidence}}}$$

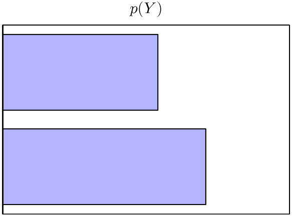
An Illustration of a Distribution over Two Variables



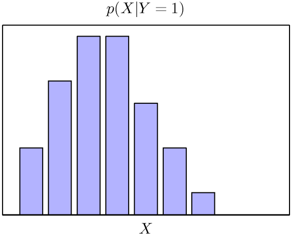
joint



marginal



marginal



conditional

Statistical Independence: Definition

Definition: Independent Variables

Two variables X and Y are statistically independent, denoted $X \perp\!\!\!\perp Y$, if and only if their joint distribution *factorizes* into the product of their marginals:

$$X \perp\!\!\!\perp Y \leftrightarrow p(X, Y) = p(X)p(Y)$$

We may also consider random variables that are conditionally independent given some other variable.

Definition: Conditionally Independent Variables

Two variables X and Y are conditionally independent given Z, denoted $X \perp\!\!\!\perp Y|Z$, if and only if

$$p(X, Y|Z) = p(X|Z)p(Y|Z)$$

Intuitively, Z is a common cause for X and Y.

Information

Say that a message comprises an answer to a single, yes/no question — *e.g.*, Will rain tomorrow or not?

Informally, the amount of information in such a message is how *unexpected* or "surprising" it is.

- If you are 90% sure it will not rain tomorrow, learning that it is raining is more suprising than if you learnt it was not raining.

Information

For X a random variable with outcomes in $\mathcal{X}$ and distribution $p(X)$ the information in learning $X = x$ is $h(x) = \log_2 \frac{1}{p(x)} = -\log_2 p(x)$.

The information in observing x is large when $p(x)$ is small and *vice versa*. Rare events are more informative.

Entropy

The entropy of a random variable X is the average information content of its outcomes.

Entropy

Let X be a discrete r.v. with possible outcomes $\mathcal{X}$ and distribution $p(X)$. The entropy of X — or, equivalently, $p(X)$ — is

$$H(X) = \mathbb{E}_X\left[h(X)\right] = -\sum_x p(x) \log_2 p(x)$$

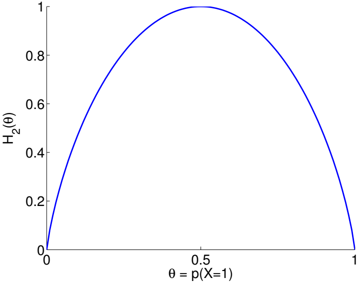
where we define $0 \log 0 \equiv 0$, as $\lim_{p \to 0} p \log p = 0$.

Example 1: $\mathcal{X} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$; $p(\mathtt{a}) = p(\mathtt{b}) = \frac{1}{8}$, $p(\mathtt{c}) = \frac{1}{4}$, $p(\mathtt{d}) = \frac{1}{2}$.
Entropy $H(X) = 2\frac{1}{8} \log_2 8 + \frac{1}{4} \log_2 4 + \frac{1}{2} \log_2 2 = 2\frac{3}{8} + \frac{2}{4} + \frac{1}{2} = 1.75$.

Example 2: $\mathcal{X} = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$; $p(\mathtt{a}) = p(\mathtt{b}) = p(\mathtt{c}) = p(\mathtt{d}) = \frac{1}{4}$.
Entropy $H(X) = 4\frac{1}{4} \log_2 4 = 2$.

Example 3 — Bernoulli Distribution

Let $X \in \{0, 1\}$ with $X \sim \text{Bern}(X|\theta)$: $p(X = 0) = 1 - \theta$ and $p(X = 1) = \theta$.
Entropy of X is $H(X) = H_2(\theta) := -\theta \log \theta - (1 - \theta) \log(1 - \theta)$.



- Minimum entropy $\rightarrow$ no uncertainty about X, i.e. $\theta = 1$ or $\theta = 0$
- Maximum when $\rightarrow$ complete uncertainty about X, i.e. $\theta = 0.5$
- For $\theta = 0.5$ (e.g. a fair coin) $H_2(X) = 1$ bit.

Property: Concavity

Proposition

Let $\mathbf{p} = (p_1, \ldots, p_N)$. The function $H(\mathbf{p}) := -\sum_{i=1}^{N} p_i \ln p_i$ is concave.

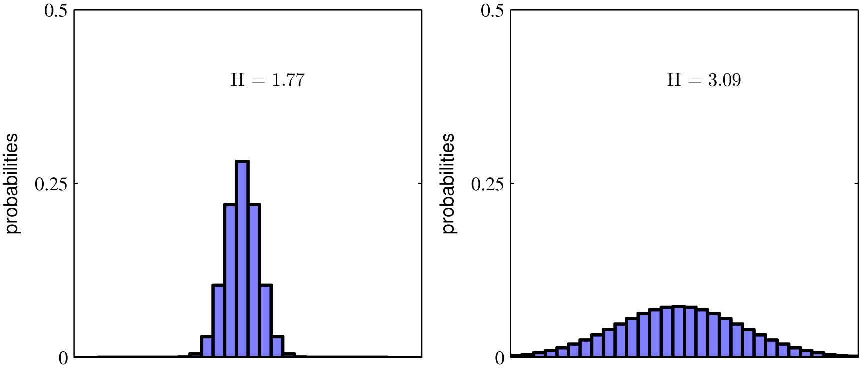
First derivative is $\nabla H(\mathbf{p}) = -\left(\ln p_1 + 1, \ldots, \ln p_N + 1\right)^\top$ and so second derivative is $\nabla^2 H(\mathbf{p}) = \operatorname{diag}\left(-p_1^{-1}, \ldots, -p_N^{-1}\right)$, which is negative semi-definite so $H(\mathbf{p})$ is concave.

We can switch between $\log_2$ and ln since for $x > 0$
$\log_2 x = \log_2 e^{\ln x} = \ln x . \log_2 e$.

When entropy is defined using $\log_2$ its *base* is 2 and units are *bits*. When entropy is defined using ln it has base e and units of *nats*.

Example 4 — Categorical Distribution

Categorical distributions with 30 different states:



(Figure from Bishop, PRML, 2006)

- The more sharply peaked the lower the entropy
- The more evenly spread the higher the entropy
- Maximum for *uniform* distribution: $H(X) = -\log \frac{1}{30} \approx 3.40$ nats
  - When will the entropy be minimum?

Property: Maximised by Uniform Distribution

Proposition

Let X take values from $\mathcal{X} = \{1, \ldots, N\}$ with distribution $\mathbf{p} = (p_1, \ldots, p_N)$ where $p_i = p(X = i)$. Then $H(X) \leq \log_2 N$ with equality iff $p_i = \frac{1}{N}\ \forall i$.

Sketch Proof:
Objective: $\max_{\mathbf{p}} H(X) = -\sum_{i=1}^{N} p_i \log p_i$ s.t. $\sum_{i=1}^{N} p_i = 1$. Lagrangian:

$$\mathcal{L}(\mathbf{p}) = -\sum_i p_i \log p_i + \lambda \left(\sum_i p_i - 1 \right). \tag{1}$$

$\nabla \mathcal{L}(\mathbf{p}) = 0$ gives $\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_i p_i - 1 = 0$ and $\frac{\partial \mathcal{L}}{\partial p_i} = -(\log p_i + 1) + \lambda = 0$ so $\log p_i = \lambda - 1 \implies p_i = 2^{\lambda - 1}$. Summing p_i gives $1 = \sum_i 2^{\lambda-1} = N.2^{\lambda-1}$. Taking logs: $0 = \log_2 N + \lambda - 1$ so $p_i = 2^{-\log_2 N} = \frac{1}{N}$.

Note that $\log_2 N$ is number of bits needed to describe an outcome of X.

Property: Decomposability

For a r.v. X on $\mathcal{X} = \{x_1, \ldots, x_N\}$ with probability distribution $\mathbf{p} = (p_1, \ldots, p_N)$:

$$H(X) = H(X^{(1)}) + (1 - p_1) H\left(X^{(2:N)}\right)$$

$X^{(1)} \in \{0, 1\}$ indicates if $X = x_1$ or not, so:
$p(X^{(1)} = 1) = p(X = x_1) = p_1$ and $p(X^{(1)} = 0) = p(X \neq x_1) = 1 - p_1$

$X^{(2:N)} \in \{x_2, \ldots, x_N\}$ is r.v. over outcomes except x_1 and
$p(X^{(2:N)} = x) = p(X = x | X \neq x_1) = \left(\frac{p_2}{1-p_1}, \ldots, \frac{p_{|\mathcal{X}|}}{1-p_1}\right)$

Joint Entropy

The joint entropy $H(X, Y)$ of a pair of discrete random variables with joint distribution $p(X, Y)$ is given by:

$$H(X, Y) = \mathbb{E}_{X,Y}\left[\log \frac{1}{p(X, Y)}\right]$$
$$= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(x, y)}$$

Easy to remember: This is just the entropy $H(Z)$ for a random variable $Z = (X, Y)$ over $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with distribution $p(Z) = p(X, Y)$.

Joint Entropy:

Independent Random Variables

If X and Y are statistically independent we have that:

$$
\begin{aligned}
H(X, Y) &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log \frac{1}{p(x, y)} \\
&= -\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x) p(y) \big[\log p(x) + \log p(y) \big] \\
&= -\sum_{x \in \mathcal{X}} p(x) \log p(x) \underbrace{\sum_{y \in \mathcal{Y}} p(y)}_{1} - \sum_{y \in \mathcal{Y}} p(y) \log p(y) \underbrace{\sum_{x \in \mathcal{X}} p(x)}_{1} \\
&= \sum_{x \in \mathcal{X}} p(x) \log \frac{1}{p(x)} + \sum_{y \in \mathcal{Y}} p(y) \log \frac{1}{p(y)} \\
&= H(X) + H(Y)
\end{aligned}
$$

Entropy is additive for independent random variables.

Also, $H(X, Y) = H(X) + H(Y)$ implies $p(X, Y) = p(X)p(Y)$.

An Axiomatic Characterisation

Why that definition of entropy? Why not another function?

Suppose we want a measure $H(X)$ of "information" in r.v. X so that

1. H depends on the distribution of X, and not the outcomes themselves
2. The H for the combination of two variables X, Y is at most the sum of the corresponding H values
3. The H for the combination of two independent variables X, Y is the sum of the corresponding H values
4. Adding outcomes with probability zero does not affect H
5. The H for an unbiased Bernoulli is 1
6. The H for a Bernoulli with parameter p tends to 0 as $p \to 0$

Then, the only possible choice for H is

$$H(X) = -\sum_{x} p(x) \log_2 p(x)$$

Conditional Entropy

The conditional entropy of Y given $X = x$ is the entropy of the probability distribution $p(Y|X = x)$:

$$H(Y|X = x) = \sum_{y \in \mathcal{Y}} p(y|X = x) \log \frac{1}{p(y|X = x)}$$

The conditional entropy of Y given X, is the average over X of the conditional entropy of Y given $X = x$:

$$\begin{aligned} H(Y|X) &= \sum_{x \in \mathcal{X}} p(x) H(Y|X = x) \\ &= \sum_{x \in \mathcal{X}} p(x) \sum_{y \in \mathcal{Y}} p(y|x) \log \frac{1}{p(y|x)} \\ &= \mathbb{E}_{X,Y}\left[\frac{1}{p(Y|X)}\right] \end{aligned}$$

Average uncertainty that remains about Y when X is known.

Chain Rule

The joint entropy can be written as:

$$\begin{aligned} H(X,Y) &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(x,y) \\ &= -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\big[\log p(x) + \log p(y|x)\big] \\ &= -\sum_{x\in\mathcal{X}} \log p(x) \underbrace{\sum_{y\in\mathcal{Y}} p(x,y)}_{p(x)} - \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p(x,y)\log p(y|x) \\ &= H(X) + H(Y|X) = H(Y) + H(X|Y) \end{aligned}$$

The joint uncertainty of X and Y is the uncertainty of X plus the uncertainty of Y given X

Relative Entropy

Definition

The relative entropy or Kullback-Leibler (KL) divergence between two probability distributions $p(X)$ and $q(X)$ is defined as:

$$D_{\mathrm{KL}}(p\|q) = \sum_{x\in\mathcal{X}} p(x) \log \frac{p(x)}{q(x)} = \mathbb{E}_{p(X)}\left[\log \frac{p(X)}{q(X)}\right].$$

- Note:
  - Both $p(X)$ and $q(X)$ are defined over the same alphabet $\mathcal{X}$
- Conventions:

$$0 \log \frac{0}{0} \stackrel{\text{def}}{=} 0 \qquad 0 \log \frac{0}{q} \stackrel{\text{def}}{=} 0 \qquad p \log \frac{p}{0} \stackrel{\text{def}}{=} \infty$$

Relative Entropy

Properties:

- $D_{\mathrm{KL}}(p \| q) \geq 0$
- $D_{\mathrm{KL}}(p \| q) = 0 \Leftrightarrow p = q$
- $D_{\mathrm{KL}}(p \| q) \neq D_{\mathrm{KL}}(q \| p)$

Observations:

- Not a true distance since is not symmetric and does not satisfy the triangle inequality
- Hence, "KL divergence" rather than "KL distance"
- Very important in machine learning and information theory. The "right" distance for distributions.

Mutual Information

Let X, Y be two r.v. with joint $p(X, Y)$ and marginals $p(X)$ and $p(Y)$:

Definition

The *mutual information* $I(X; Y)$ is the relative entropy between the joint distribution $p(X, Y)$ and the product distribution $p(X)p(Y)$:

$$\begin{aligned} I(X;Y) &= D_{\mathrm{KL}}\left(p(X,Y) \| p(X)p(Y)\right) \\ &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \end{aligned}$$

Measures "how far away" the joint distribution is from independent.

Intuitively, how much information, on average, does X convey about Y.

Relationship between Entropy and Mutual Information

We can re-write the definition of mutual information as:

$$
\begin{aligned}
I(X;Y) &= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} \\
&= \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log \frac{p(x|y)}{p(x)} \\
&= -\sum_{x \in \mathcal{X}} \log p(x) \sum_{y \in \mathcal{Y}} p(x,y) - \left(-\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x,y) \log p(x|y) \right) \\
&= H(X) - H(X|Y)
\end{aligned}
$$

The average reduction in uncertainty of X due to the knowledge of Y.

Mutual Information:
Properties

- Mutual Information is non-negative:

$$I(X;Y) \geq 0$$

- Since $H(X,Y) = H(X) + H(Y|X)$ we have that:

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

- Above is symmetric in X and Y so

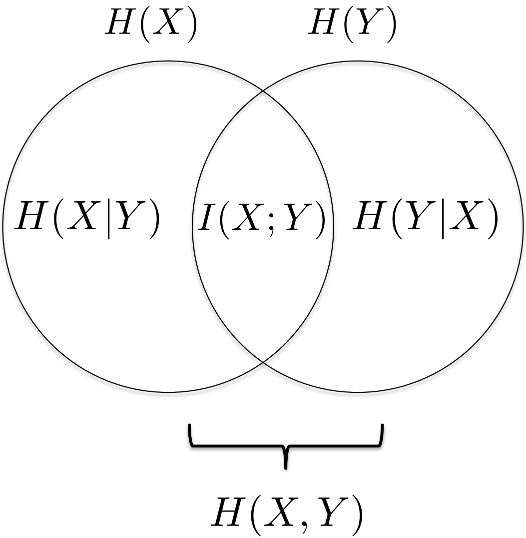
$$I(X;Y) = I(Y;X)$$

- Finally:

$$I(X;X) = H(X) - H(X|X) = H(X)$$

Sometimes the entropy is referred to as *self-information*

Breakdown of Joint Entropy



Conditional Mutual Information

The conditional mutual information between X and Y given $Z = z_k$:

$$I(X; Y|Z = z_k) = H(X|Z = z_k) - H(X|Y, Z = z_k).$$

Averaging over Z we obtain:

The conditional mutual information between X and Y given Z:

$$\begin{aligned} I(X; Y|Z) &= H(X|Z) - H(X|Y, Z) \\ &= \mathbb{E}_{p(X,Y,Z)} \log \frac{p(X, Y|Z)}{p(X|Z)p(Y|Z)} \end{aligned}$$

Summary & Next Time

Summary:

- Probability (Joint, Marginal, Conditional, Dependence)
- Information, Entropy (Joint, Conditional) & Properties
- Relative Entropy & (Conditional) Mutual Information

Next Time:

- Probabilistic Inequalities (Markov, Chebyshev)
- Information Theoretic Inequalities (Gibbs, Kraft, Data Processing)
- Source Coding Theorems
- Noisy-Channel Coding Theorem

Questions?